Vieta's formulas

François Viète
(1540-1603)

François Viète was a french lawyer and mathematician. He worked on trigonometry and on equations. One of his merits is the introduction of letters for general numbers.

Vieta's fomulas:

x_1 and x_2 are solutions of the equation

$$x^2 + p\,x + q = 0$$

if and only if

$$x_1 + x_2 = -\,p \quad \text{and} \quad x_1 \cdot x_2 = q$$

The formulas of Vieta are mostly used for checks.

Vieta's formulas: Proof

monic form (p-q-form): $x^2 + px + q = 0$

$$x_1 = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}\,, \qquad x_2 = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x_1 + x_2 = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q} - \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q} = -p$$

$$x_1 \cdot x_2 = \left(-\frac{p}{2}\right)^2 - \left(\sqrt{\left(\frac{p}{2}\right)^2 - q}\right)^2 = \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q = q$$

(a-b-c-form): $a x^2 + b x + c = 0 \quad \Leftrightarrow \quad x^2 + \frac{b}{a} x + \frac{c}{a} = 0$

$$x_1 + x_2 = -\frac{b}{a}\,, \qquad x_1 \cdot x_2 = \frac{c}{a}$$

Quadratic equations: Exercises 7-10

Find a quadratic equation with given solutions

Exercise 7: $x_1 = -3, \quad x_2 = 5$

Exercise 8: $x_1 = 3 - \sqrt{2}, \quad x_2 = 3 + \sqrt{2}$

x_1 and x_2 are solutions of the quadratic equation G. Determine the following expressions without solving for x_1 and x_2

Exercise 9: $G: \quad x^2 + 2x - 14 = 0$

$$\frac{3 - 5x_1}{x_1 + x_2} + \frac{3 - 5x_2}{x_1 + x_2}$$

Exercise 10: $G: \quad x^2 - 20x + 8 = 0$

$$\frac{5 + 2x_1}{x_1 + x_2} + \frac{5 + 2x_2}{x_1 + x_2}$$

$$x^2 + px + q = 0, \qquad x_1 + x_2 = -p, \qquad x_1 \cdot x_2 = q$$

Solution 7: $x_1 = -3, \qquad x_2 = 5$

$$p = -(x_1 + x_2) = -2$$

$$q = x_1 \cdot x_2 = -15$$

quadratic equation: $x^2 - 2x - 15 = 0$

Solution 8: $x_1 = 3 - \sqrt{2}, \qquad x_2 = 3 + \sqrt{2}$

$$p = -(x_1 + x_2) = -6$$

$$q = x_1 \cdot x_2 = (3 - \sqrt{2})(3 + \sqrt{2}) =$$

$$= 3^2 - (\sqrt{2})^2 = 9 - 2 = 7$$

quadratic equation: $x^2 - 6x + 7 = 0$

Solution 9:

$$G: \quad x^2 + 2x - 14 = 0 \quad \Rightarrow \quad p = 2, \quad q = -14$$

$$D = 1^2 - (-14) = 15 > 0 \quad \Rightarrow \quad x_1 \neq x_2$$

$$x_1 + x_2 = -p = -2, \quad x_1 \cdot x_2 = q = -14$$

$$\frac{3 - 5x_1}{x_1 + x_2} + \frac{3 - 5x_2}{x_1 + x_2} = \frac{6 - 5(x_1 + x_2)}{x_1 + x_2} = -5 + \frac{6}{x_1 + x_2} =$$

$$= -5 - 3 = -8$$

Solution 10:

$$G: \quad x^2 - 20x + 8 = 0 \quad \Rightarrow \quad p = -20, \quad q = 8$$

$$D = (-20)^2 - 4\,8 = 368 > 0 \quad \Rightarrow \quad x_1 \neq x_2$$

$$x_1 + x_2 = -p = 20, \quad x_1 \cdot x_2 = q = 8$$

$$\frac{5 + 2x_1}{x_1 + x_2} + \frac{5 + 2x_2}{x_1 + x_2} = \frac{10 + 2(x_1 + x_2)}{x_1 + x_2} = 2 + \frac{10}{x_1 + x_2} =$$

$$= 2 + \frac{1}{2} = \frac{5}{2}$$

Quadratic equations: Exercises 11-13

u and v are solutions of the quadratic equation G. Determine the following expressions without determination of u and v.

Exercise 11: $G: \quad 5x^2 + 2x - 4 = 0$

$$\frac{5 - 2u}{u} + \frac{5 + 4v}{v}$$

Exercise 12: $G: \quad 3x^2 - 5x - 4 = 0$

$$\frac{3 + 5u}{u} + \frac{3 + 4v}{v}$$

Exercise 13: $G: \quad x^2 - 2x - 5 = 0$

$$\frac{vu^3 - uv^3}{v - u}$$

Solution 11:

$$G: \quad 5x^2 + 2x - 4 = 0 \quad \Leftrightarrow \quad x^2 + \frac{2}{5}x - \frac{4}{5} = 0$$

$$p = \frac{2}{5}, \quad q = -\frac{4}{5}$$

$$D = 2^2 - 4 \cdot 5 \cdot (-4) = 84 > 0, \quad u \neq v$$

$$u + v = -p = -\frac{2}{5}, \quad u \cdot v = q = -\frac{4}{5}$$

$$\frac{5 - 2u}{u} + \frac{5 + 4v}{v} = \frac{(5 - 2u)v}{uv} + \frac{(5 + 4v)u}{uv} =$$

$$= \frac{5(u + v) + 2uv}{uv} = 2 + \frac{5}{2} = \frac{9}{2}$$

Solution 12: $\quad \dfrac{3 + 5u}{u} + \dfrac{3 + 4v}{v} = \dfrac{21}{4}$

Solution 13: $\quad \dfrac{vu^3 - uv^3}{v - u} = -uv\,(u + v) = 10$

$$a\,x^2 + b\,x + c \;=\; a\left(x^2 + \frac{b}{a}\,x + \frac{c}{a}\right) =$$

$$x_1 + x_2 = -\frac{b}{a}\,, \qquad x_1 \cdot x_2 = \frac{c}{a}$$

$$= a\left[\,x^2 - (x_1 + x_2)\;x + x_1 \cdot x_2\,\right] =$$

$$= a\left[\,(x^2 - x_1\,x) - (x_2\,x - x_1\,x_2)\,\right] =$$

$$= a\left[\,x\,(x - x_1) - x_2(x - x_1)\,\right] =$$

$$= a\;(x - x_1)\;(x - x_2)$$

Every algebraic equation of the form $a\,x^2 + bx + c = 0$ can be factorised as follows

$$\boxed{a\,x^2 + b\,x + c \;=\; a\;(x - x_1)\;(x - x_2)}$$

Quadratic equations: Exercises 14-15

Determine the solutions of the following quadratic equations

Exercise 14: $$a + 2b = \frac{x^2 - 4bx}{a - 2b}$$

Exercise 15: $$x(x + 3) + a(a - 3) = 2(ax - 1)$$

$$a + 2b = \frac{x^2 - 4bx}{a - 2b}$$

$$a + 2b = \frac{x^2 - 4bx}{a - 2b} \quad | \quad \times\ (a - 2b) \quad (a \neq 2b)$$

$$(a + 2b)(a - 2b) = x^2 - 4bx \quad \Leftrightarrow \quad x^2 - 4bx + 4b^2 - a^2 = 0$$

$$x^2 + px + q = 0, \qquad p = -4b, \qquad q = 4b^2 - a^2$$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} = 2b \pm \sqrt{(-2b)^2 - 4b^2 + a^2}$$

$$x_{1,2} = 2b \pm \sqrt{a^2} \quad \Rightarrow \quad x_1 = 2b + a, \qquad x_2 = 2b - a$$

$$x(x+3)+a(a-3)=2(ax-1)$$

$$x^2+(3-2a)x+a(a-3)+2=0$$

$$x_{1,2}=-\frac{(3-2a)}{2}\pm\sqrt{\left(\frac{3-2a}{2}\right)^2-a(a-3)-2}=$$

$$=\frac{2a-3}{2}\pm\sqrt{\frac{1}{4}}=a-\frac{3}{2}\pm\frac{1}{2}$$

$$x_1=a-2,\qquad x_2=a-1$$