Even, odd or neither? Algebraic and graphical proof
Symmetry of a function: Exercise 3

Determine algebraically and graphically whether the functions are even, odd or neither:

\[ a) \quad f(x) = x^4 - 2x^2, \quad b) \quad f(x) = x^3 - 4x. \]

Comment:

**Algebraic proof:** transform \( x \) to \(-x\) and compare \( f(x) \) to \( f(-x) \).

**Graphical proof:** decide on the symmetry properties by visual inspection of the function graph.
Algebraic solution $a$):

To check whether a function is even, odd or neither, we first have to find $f(-x)$ and then to decide which of the following equations holds:

(1): $f(-x) = f(x)$,  
(2): $f(-x) = -f(x)$

If equation (1) is true, it is an even function, if equation (2), it is an odd function. If neither of them holds, the function is neither even nor odd.

$$f(x) = x^4 - 2x^2$$

$$f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$$

Raising (-x) to an even power $n$, is just the same as raising $+x$ to the power $n$:

$$(-x)^2 = (-x)\cdot(-x) = x^2$$

$$(-x)^4 = (-x)\cdot(-x)\cdot(-x)\cdot(-x) = x^4$$

Graphical solution: to make a statement on the function properties, in the present case on the symmetries, by looking at the graph.
Example 1: Graphical solution 3a

Fig. 3-1: The graph of $y = f(x)$ (a) is symmetric with respect to the y-axis. The function is even.
Example 1: Algebraic solution 3b

**Algebraic solution:**

\[ f(x) = x^3 - 4x \]

\[ f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x = -(x^3 - 4x) = -f(x) \]

Raising \((-x)\) to an **odd** power \(n\) yields the negative of raising \(+x\) to the power \(n\). The function contains two odd powers, 1 and 3, of \(-x\).

\[ (-x)^3 = (-x) \cdot (-x) \cdot (-x) = -x^3 \]
Fig. 3-2: The graph of $y = f(x)$ (b) is symmetric with respect to the origin. The function is odd.
Determine, whether a function \( y = f(x) \) is even, odd or neither in the given domains:

\[
f(x) = \frac{x^2}{2} - 2, \quad a) \ D_f = \mathbb{R}, \quad b) \ D_f = [-2, 3]
\]

Explain how symmetry properties can be influenced by changing the function domain.
Algebraic proof:

\[ f(x) = \frac{x^2}{2} - 2 \]

\[ f(-x) = \frac{(-x)^2}{2} - 2 = \frac{x^2}{2} - 2 = f(x) \]

The algebraic check of the function term indicates, that the function is even. However this algebraic check is not enough. The following figures 4-1 and 4-2 show, that also the function domain can influence the symmetry.

a) \( D_f = \mathbb{R} \)

Here we have a symmetric domain, the function graph (Fig. 4-1) is symmetric with respect to the \( y \)-axis.

b) \( D_f = [-2, 3] \)

In this case the function graph (Fig. 4-2) is not symmetric with respect to the \( y \)-axis, as the domain is not symmetric. To point (3, 2.5) there is no symmetric point (-3, 2.5) on the function graph.
Fig. 4-1: The graph of the function $y = f(x)$ is symmetric with respect to the $y$-axis.
Symmetry of a function: Solution 4

Fig. 4-2: The graph of the function \( y = f(x) \) is not symmetric with respect the y-axis.
Exercise 5: Determine

1) which of the given functions are polynomial functions

\( a \) \( f(x) = x^2 - 2 \)

\( b \) \( f(x) = x^3 - 1 \)

\( c \) \( f(x) = 5x^2 - 3x + \sqrt{x} \)

\( d \) \( f(x) = x^2 - 2x - 1 \)

\( e \) \( f(x) = -x^{5/2} + 7x^2 - 11x \)

\( g \) \( f(x) = x^5 - 4x^3 + x \)

2) which of the polynomial functions are even, odd or neither.

Exercise 6:

Formulate the condition for a given polynomial function, to be even odd or neither of them.
1) A **polynomial** \( y = f(x) \) is an expression constructed from variables and constants, using only the operations of addition, subtraction and multiplication. A polynomial can have non-negative integer exponents only. The polynomial functions of this exercise are:

\[

text{a)} \quad f(x) = x^2 - 2 \\

text{b)} \quad f(x) = x^3 - 1 \\

text{d)} \quad f(x) = x^2 - 2x - 1 \\

text{g)} \quad f(x) = x^5 - 4x^3 + x
\]

They have integer exponents only. The functions \( c) \) and \( e) \) are no polynomials. The last term of \( c) \) is \( x \) raised to power \( 1/2 \) and the first term of \( e) \) is \( x \) raised to \( 5/2 \).

\[

text{c)} \quad f(x) = 5x^2 - 3x + \sqrt{x} \\

text{e)} \quad f(x) = -x^{5/2} + 7x^2 - 11x
\]
2) To determine which of the polynomial functions is even, odd or neither, we find for each function $f(-x)$.

\[ a) \quad f(x) = x^2 - 2, \quad f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x) \]

\[ b) \quad f(x) = x^3 - 1, \quad f(-x) = (-x)^3 - 1 = -x^3 - 1 \neq f(x) \]

\[ d) \quad f(x) = x^2 - 2x - 1, \quad \]

\[ f(-x) = (-x)^2 - 2(-x) - 1 = x^2 + 2x - 1 \neq f(x) \]

\[ g) \quad f(x) = x^5 - 4x^3 + x, \quad \]

\[ f(-x) = (-x)^5 - 4(-x)^3 + (-x) = -x^5 + 4x^3 - x = \]

\[ = -(x^5 - 4x^3 + x) = -f(x) \]

The function \(a)\) is even, the function \(g)\) is odd, and the functions \(b)\) and \(d)\) are neither even nor odd.
There is also another possibility to test, whether a function is even, odd or neither. As we have shown algebraically the function \( a \) is even:

\[
a) \quad f(x) = x^2 - 2, \quad f(-x) = f(x)
\]

Let us take two values of \( x \), \( x = 1 \) and \( x = -1 \), which are symmetric about the origin and evaluate the function for both \( x \)-values:

\[
f(-1) = (-1)^2 - 2 = 1 - 2 = -1, \quad f(1) = 1^2 - 2 = 1 - 2 = -1
\]

\[
f(-1) = f(1)
\]

\[
b) \quad f(x) = x^3 - 1, \quad f(-x) \neq f(x)
\]

\[
f(-1) = (-1)^3 - 1 = -1 - 1 = -2, \quad f(1) = 1^3 - 1 = 1 - 1 = 0
\]

\[
f(-1) \neq \pm f(1)
\]
Function symmetry: Solution 5

\[ d) \quad f(x) = x^2 - 2x - 1, \quad f(-x) \neq f(x) \]

\[ f(-1) = (-1)^2 - 2 \cdot (-1) - 1 = 1 + 2 - 1 = 2 \]

\[ f(1) = 1^2 - 2 \cdot 1 - 1 = 1 - 2 - 1 = -2 \]

\[ f(-1) = -f(1) \]

We have shown already, that this function is neither even nor odd. However, for these particular \( x \)-values the function shows the property of being odd.

To remember!

If we just evaluate the function at two \( x \)-values which are symmetric about the origin, we have to be sure, that the same result will be obtained for all other pair of symmetric \( x \)-values of the function domain.

The figure on the next page shows the graph of this function with two points:

\[ P_1 = (1, f(1)) = (1, -2), \quad P_2 = (-1, f(-1)) = (-1, 2) \]
Fig. 4: Graph of a neither even nor odd function $y = f(x)$ with two points.
Function symmetry: Solution 6

\((-x)^n = \begin{cases} 
  x^n, & \text{if } n \text{ is even} \\
  -x^n, & \text{if } n \text{ is odd}
\end{cases}\)

Terms with \textit{even} powers of \(x\) will remain the same, when \(x\) is replaced by \(-x\).

Terms with \textit{odd} powers of \(x\) will change sign, when \(x\) is replaced by \(-x\).
Polynomial functions with terms containing only even powers of the variable $x$ and multiple or additive constants are even functions. For example, the following functions are even:

$$f(x) = \frac{x^2}{2} - 2$$

$$g(x) = -3x^4 + 6x^2 - 1$$

$$h(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

These functions are presented in Fig. 5-1.

A multiple constant is a numerical or constant factor in an algebraic term. For example, $1/2$ is the multiple constant in the function $f(x)$. -3 and 6 are multiple constants in the function $g(x)$.

The constant -2 in the function $f(x)$ and -1 in the function $g(x)$ are additive constants.
Function symmetry: Solution 6

Fig. 5-1: Graphs of even functions
Polynomial functions with terms containing **only** odd powers of the variable $x$ are odd functions. For example, **odd** functions are:

$$f(x) = \frac{x^3}{6}$$

$$g(x) = -6x^5 + 9x^3 - x$$

$$h(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$
Fig. 5-2: Graphs of odd functions
Function symmetry: Exercise 7

Determine which of the functions given below are even, odd or neither:

a) \( f(x) = \frac{1}{x} \), \( g(x) = \frac{1}{x - 3} \), \( h(x) = \frac{2x}{x + 7} \)

b) \( f(x) = \frac{1}{x^2} \), \( g(x) = \frac{x}{x^2 + 1} \), \( h(x) = \frac{5x^3}{x^2 - 16} \)

c) \( f(x) = \frac{1}{x^3} \), \( g(x) = \frac{3}{x^3 - 4x} \), \( h(x) = \frac{2x^3 - x^2}{x^2 + 5} \)

d) \( f(x) = \frac{x^2 + 7}{x^2 - 3x^4} \), \( g(x) = \frac{x^3 - 11x}{x^4 + 12} \), \( h(x) = \frac{5x^3}{x^7 - 9x^3} \)

e) \( f(x) = \frac{1}{|x|} \), \( g(x) = \frac{|x|}{x} \), \( h(x) = \frac{1}{|x| + 1/2} \)

e) \( f(x) = \frac{1}{x^2 + 2|x| + 1} \), \( g(x) = \frac{1}{x^2 - 0.8|x| + 1/2} \)

Formulate the conditions for a rational function to be even or odd.
Definition:

A **rational function** is a function which can be defined by a rational fraction, i.e. an algebraic fraction such that both the numerator and the denominator are polynomials.

\[
f(x) = \frac{P(x)}{Q(x)}, \quad Q(x) \neq 0
\]

Example of functions \( b) \):

\[
f(x) = \frac{1}{x^2}, \quad P_f(x) = 1, \quad Q_f(x) = x^2
\]

\[
g(x) = \frac{x}{x^2 + 1}, \quad P_g(x) = x, \quad Q_g(x) = x^2 + 1
\]

\[
h(x) = \frac{5x^3}{x^2 - 16}, \quad P_h(x) = 5x^3, \quad Q_h(x) = x^2 - 16
\]
Function symmetry: Solution 7 a,b

\(a\) \( f(x) = \frac{1}{x} \), \( g(x) = \frac{1}{x - 3} \), \( h(x) = \frac{2x}{x + 7} \)

\[ f(-x) = -\frac{1}{x} = -f(x), \quad g(-x) = \frac{1}{-x - 3} = -\frac{1}{x + 3} \]

\[ h(-x) = -\frac{2x}{-x + 7} = \frac{2x}{x - 7} \]

The function \( f(x) \) is odd.

\(b\) \( f(x) = \frac{1}{x^2} \), \( g(x) = \frac{x}{x^2 + 1} \), \( h(x) = \frac{5x^3}{x^2 - 16} \)

\[ f(-x) = \frac{1}{x^2} = f(x), \quad g(-x) = \frac{(-x)}{(-x)^2 + 1} = -\frac{x}{x^2 + 1} = -g(x) \]

\[ h(-x) = \frac{5(-x)^3}{(-x)^2 - 16} = -\frac{5x^3}{x^2 - 16} = -h(x) \]

The function \( f(x) \) is even, the functions \( g(x) \) and \( h(x) \) are odd.
The function \( f(x) \) and \( g(x) \) are odd.
Function symmetry: Solution 7d

\( d \quad f(x) = \frac{x^2 + 7}{x^2 - 3x^4}, \quad g(x) = \frac{x^3 - 11x}{x^4 + 12}, \quad h(x) = \frac{5x^3}{x^7 - 9x^3} \)

\[
\begin{align*}
    f(-x) &= \frac{(-x)^2 + 7}{(-x)^2 - 3(-x)^4} = \frac{x^2 + 7}{x^2 - 3x^4} = f(x) \\
    g(-x) &= \frac{(-x)^3 - 11 \cdot (-x)}{(-x)^4 + 12} = \frac{-x^3 + 11x}{x^4 + 12} = -\frac{x^3 - 11x}{x^4 + 12} = -g(x) \\
    h(-x) &= \frac{5 \cdot (-x)^3}{(-x)^7 - 9(-x)^3} = \frac{5x^3}{x^7 - 9x^3} = h(x)
\end{align*}
\]

The functions \( f(x) \) and \( h(x) \) are even, the function \( g(x) \) is odd.
Function symmetry: Solution 7e

Fig. 7-1: Graph of the even function $y = f(x)$

$f(x) = \frac{1}{|x|}, \quad f(-x) = \frac{1}{|-x|} = \frac{1}{|x|} = f(x)$
Function symmetry: Solution 7e

Fig. 7-2: Graph of the odd function $y = g(x)$

$$g(x) = \frac{|x|}{x}, \quad g(-x) = \frac{|-x|}{-x} = -\frac{|x|}{x} = -g(x)$$
Fig. 7-3: Graph of the even function \( y = h(x) \)

\[
h(x) = \frac{1}{|x| + 1/2}, \quad h(-x) = \frac{1}{|-x| + 1/2} = \frac{1}{|x| + 1/2} = h(x)
\]
Fig. 7-4: Even functions, which we may observe (Lüneburg)
Fig. 7-4: Graphs of even functions

\[
\begin{align*}
  f(x) &= \frac{1}{x^2 + 2|x| + 1}, \\
  g(x) &= \frac{1}{x^2 - 0.8|x| + 1/2}
\end{align*}
\]
A rational function $y = f(x)$ is even, when

- the numerator $P(x)$ and the denominator $Q(x)$ are even functions, for example, $y = f(x) \ a)$ and $y = f(x) \ d)$:

  
  $a) \quad f(x) = \frac{1}{x^2}, \quad d) \quad f(x) = \frac{x^2 + 7}{x^2 - 3x^4}, \quad e) \quad f(x) = \frac{1}{|x|}$

- the numerator $P(x) \ \text{and} \ \text{the denominator} \ Q(x)$ are odd functions, for example, $y = h(x) \ d)$:

  
  $h(x) = \frac{5x^3}{x^7 - 9x^3}$
A rational function \( y = f(x) \) is odd, when

- the numerator \( P(x) \) is odd and the denominator \( Q(x) \) is even, for example:
  
  \[
  b) \quad g(x) = \frac{x}{x^2 + 1}, \quad h(x) = \frac{5x^3}{x^2 - 16}
  \]

- the numerator \( P(x) \) is even and the denominator \( Q(x) \) is odd, for example:
  
  \[
  a) \quad f(x) = \frac{1}{x}, \quad c) \quad g(x) = \frac{3}{x^3 - 4x}, \quad e) \quad g(x) = \frac{|x|}{x}
  \]