Symmetries. Even and odd functions

Humans like to admire symmetry and are attracted to it.
What should we know

- Definitions of
  - a function,
  - a relation,
  - a function domain.

- Vertical line test.
What shall we study

• Three types of curve symmetry:
  − symmetry with respect to the $y$-axis,
  − symmetry with respect to the $x$-axis,
  − symmetry with respect to the coordinate origin.

• which of the symmetry applies to functions and which to relations,

• how the symmetry with respect to the $y$-axis and to the coordinate origin is reflected in the function equation,

• algebraical and graphical proof of the axis or origin symmetry,

• symmetry rules for some functions: polynomials, rational, trigonometric and composed functions,

• to present a function as a sum of even and odd functions.
**Exercise 1:**

In Figure 1-1 three graphs, which correspond to the following equations

\[ a) \quad y = x^2, \quad b) \quad y = \frac{x^3}{8}, \quad c) \quad x = y^2 \]

are given. Determine whether each graph is symmetric or not and describe the type of symmetry.
Symmetry of a graph: Exercise 1a

Fig. 1-1a: Graph of the function $f(x) = x^2$
Symmetry of a graph: Exercise 1b

Fig. 1-1b: Graph of the function $f(x) = \frac{x^3}{8}$
Symmetry of a graph: Exercise 1c

Fig. 1-1c: Graph of the equation $x = y^2$
Symmetry of a graph: Solution 1a

\( a \) \( y = x^2 \)

The graph of the function in Fig. 1-1a is symmetric with respect to the \( y \)-axis. It means, that for each point \((x, y) = (x, f(x))\) on the graph there is the point \((-x, y) = (-x, f(x))\) on the same graph:

\[
\begin{align*}
(x, f(x)) & \rightarrow (-x, f(-x)) \\
(1, 1) & \rightarrow (-1, 1) \\
(2, 4) & \rightarrow (-2, 4)
\end{align*}
\]

Algebraical expression of the symmetry with respect to the \( y \)-axis:

For all \( x \) of the function domain the symmetry with respect to the \( y \)-axis means algebraically:

\[
f(-x) = f(x)
\]
Symmetry of a graph: Solution 1a

Fig. 1-2a: The graph of the function $f(x) = x^2$ is symmetric with respect to the y-axis
Symmetry of a graph: Solution 1b

\[ b \) \  y = \frac{x^3}{8} \]

The graph of the function in Fig. 1-1b is symmetric about the origin. It means, that for each point \((x, y) = (x, f(x))\) on the graph there is the point \((-x, -y) = (-x, -f(x))\) on the same graph:

\[
\begin{align*}
(x, f(x)) & \overset{O}{\rightarrow} (-x, -f(x)) = (-x, f(-x)) \\
(2, 1) & \overset{O}{\rightarrow} (-2, -1) \\
(3, 3.38) & \overset{O}{\rightarrow} (-3, -3.38)
\end{align*}
\]

*Algebraical* expression of the symmetry with respect to the origin:

For all \(x\) of the function domain the symmetry with respect to the origin means algebraically:

\[
f(-x) = -f(x)
\]
Symmetry of a graph: Solution 1b

Fig. 1-2b: The graph of the function \( f(x) = \frac{x^3}{8} \) is symmetric with respect to the point \( O(0, 0) \)
c) \( x = y^2 \)

The graph of \( x = y^2 \) in Fig. 1-1c is symmetric about the \( x \)-axis. It means, that for each point \((x, y) = (x, f(x))\) on the graph there is a point \((x, -y) = (x, -f(x))\) on the same graph:

\[
\begin{align*}
(x, f(x)) & \rightarrow (x, -f(x)) = (x, -f(x)) \\
(1, 1) & \rightarrow (1, -1) \\
(4, 2) & \rightarrow (4, -2)
\end{align*}
\]

\( x = y^2 \) is not a function but a relation. The symmetry with respect to the \( x \)-axis means that one value of \( x \) can correspond to two or more values of \( y \).
Symmetry of a graph: Solution 1c

Fig. 1-2c: The graph of the relation $x = y^2$ is symmetric with respect to the x-axis.
**Definition:**

A function \( y = f(x) \) with symmetric domain \( D \) is **even**, if the following condition holds for all \( x \) of the domain:

\[
f(-x) = f(x)
\]

The graph of an **even function** is axially symmetric with respect to the \( y \)-axis.

**This means:**

- The function graph remains unchanged after being reflected about the \( y \)-axis.
- If a function graph has a point \((x, f(x))\), it has also the point \((-x, f(x))\). This means that the domain of an even function is symmetric about the origin.
even function
Odd function

Definition:

A function \( y = f(x) \) with symmetric domain \( D \) is odd, if the following condition holds for all \( x \) of the domain:

\[
f( -x ) = -f(x)
\]

The graph of an odd function is symmetric with respect to the origin. The symmetry with respect to the origin is a rotational symmetry.

This means that:

- The graph remains unchanged after 180 degree rotation about the origin. If one rotates a right hand side of a curve by 180° about the origin, then one gets the left side of the curve.

- if a function graph has a point \((x, f(x))\), it has also the point \((-x, -f(x))\). This means that the domain of an odd function is symmetric about the origin.
The left hand part of the odd function graph can be obtained by reflecting the right hand part of the graph about the $y$-axis, followed by the reflection about the $x$-axis, as shown in Fig. 1-3c.
odd function
Do not confuse even/odd functions and even/odd integers!
Exercise 2:

In Figures 2i (i stands for the letters from a till l) the graphs are given. For each graph determine

1) a symmetry with respect to the axis or to the origin,
2) which graphs describe a function,
3) which functions are even, odd or neither,
4) which graphs describe a relation.
Symmetry of a graph: Exercise 2

Fig. 2a: Graph a)
Symmetry of a graph: Exercise 2

Fig. 2b: Graph b)
Symmetry of a graph: Exercise 2

Fig. 2c: Graph c)

$g_c$
Symmetry of a graph: Exercise 2

Fig. 2d: Graph d)
Symmetry of a graph: Exercise 2

Fig. 2e: Graph e)
Symmetry of a graph: Exercise 2

Fig. 2f: Graph $f$)
Symmetry of a graph: Exercise 2

Fig. 2g: Graph g)
Symmetry of a graph: Exercise 2

Fig. 2h: Graph h)
Symmetry of a graph: Exercise 2

Fig. 2i: Graph i)

$g_i$
Symmetry of a graph: Exercise 2

Fig. 2j: Graph j)
Symmetry of a graph: Exercise 2

Fig. 2k: Graph $k$)
Symmetry of a graph: Exercise 2

Fig. 2l: Graph 1)
Symmetry of a graph: Solution 2

- The graphs $h)$ and $j)$ are symmetric with respect to the $x$-axis.
- The graphs $a), \, d), \, g)$ and $h)$ are symmetric with respect to the $y$-axis.
- The graphs $c), \, e), \, h)$ and $k)$ are symmetric with respect to the origin.
- The graphs of $a), \, b), \, c), \, d), \, e), \, f), \, g), \, i), \, k)$ and $l)$ describe functions. This can be shown by a **vertical line** test (see Figs. 3-2 and 3-5 which present this test for graphs $f)$ and $h)$).
- The functions $a), \, d)$ and $g)$ are **even** (see Fig. 3-3).
- The functions $c), \, e)$ and $k)$ are **odd** (see Fig. 3-4).
- The graphs of $h)$ and $j)$ represent relations.
Symmetry of a graph: Solution 2

Fig. 3-2: The use of a vertical line test as a visual proof whether a graph describes a function or a relation.
The graphs a), d) and g) represent functions, which are symmetric with respect to the y-axis, they are even functions.
The graphen c), e) and k) represent functions, which are symmetric with respect to the origin, they are **odd** functions.

*Fig. 3-4: The odd functions c), e) and k)*
Fig. 3-5: Use of the vertical line test as a visual proof whether a graph describes a function or a relation