Vektoranalysis

Aufgaben mit Lösungen

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1. Ebene und räumlich Kurven

1.1. Differentiation eines Vektors nach einem Parameter

A1 Bestimmen Sie die erste und zweite Ableitung folgender Vektorfunktionen

$$a)\quad r(\vec{t}) = \begin{pmatrix} t+t^2 \\ t^3 \\ \sqrt{t} \end{pmatrix}, \qquad b)\quad r(\vec{t}) = \begin{pmatrix} t\cos t \\ t^2 \sin t \\ \sqrt[3]{t} \end{pmatrix}, \qquad c)\quad r(\vec{t}) = \begin{pmatrix} e^{2t} \\ e^{t^2} \\ t \end{pmatrix}$$

2. Gradient, Richtungsableitung

2.1. Gradient eines Skalarfeldes

Gesucht ist der Gradient von folgenden skalaren Funktionen:

A2

$$a)\quad \varphi(x,y) = 2x^2 - xy, \qquad b)\quad \varphi(x,y) = \frac{x}{y}, \qquad c)\quad \varphi(x,y) = \frac{y}{x^2+y^2},$$

$$d)\quad \varphi(x,y) = \sqrt{x^2+y^2}, \qquad e)\quad \varphi(x,y,z) = \sqrt{x^2+y^2+z^2}$$

A3

$$a)\quad \varphi(x,y) = x\,e^y - y\,e^x, \qquad b)\quad \varphi(x,y) = x\,e^{2y}, \qquad c)\quad \varphi(x,y) = y\,e^{x^2}$$

$$d)\quad \varphi(x,y) = e^{x-y} + e^{y-x}, \qquad e)\quad \varphi(x,y,z) = e^{xyz}, \qquad f)\quad \varphi(x,y,z) = x\,y^2\,e^z$$

$$g)\quad \varphi(x,y,z) = x\,e^{y-z}, \qquad h)\quad \varphi(x,y,z) = x\,e^{z^2-y}$$

A4

$$a)\quad \varphi(x,y) = \sin\,x + \cos(2y), \qquad b)\quad \varphi(x,y) = x\,\sin(y^2-3), \qquad c)\quad \varphi(x,y) = \cos\left(\frac{x}{y}\right)$$

$$d)\quad \varphi(x,y,z) = \sin\,(xyz), \qquad e)\quad \varphi(x,y,z) = \cos\left(\frac{xy}{z}\right) \qquad f)\quad \varphi(x,y,z) = x\,\ln(y+\cos\,z)$$

A5

$$a)\quad \varphi(x,y) = x\,\ln(2x-y), \qquad b)\quad \varphi(x,y,z) = x\,\ln\,y - z\,\ln\,x, \qquad c)\quad \varphi(x,y,z) = \ln(x+y+z)$$

A6 Bestimmen Sie die Richtung des größten Anstiegs der Funktion φ im Punkt P und die Steigung in dieser Richtung

$$a)\quad \varphi(x,y) = x^2 + y^2, \qquad P = (1,\ 1)$$

$$b)\quad \varphi(x,y) = x^2 + y\,e^{x-1}, \qquad P = (1,\ 2)$$

$$c)\quad \varphi(x,y,z) = xz + e^y - \sqrt{z}, \qquad P = (-1,\ 0,\ 4)$$

$$d)\quad \varphi(x,y,z) = e^{x-y} + e^{x-z} + e^{y-z}, \qquad P = (2,\ 2,\ 2), \qquad P = (1,\ 1,\ 1)$$

2.2. Richtungsableitung

A7 Bestimmen Sie die Richtungsableitung eines skalaren Feldes φ im Punkt P in Richtung des Vektors $\vec{a}$

Beispiel:

$$\varphi(x,y) = x^3 - 2\,x \cdot y^2, \qquad \vec{a} = (-1,3), \qquad P = (1,-2)$$

$$\nabla\varphi(x,y) = \left(3x^2 - 2\,y^2\right) \cdot \vec{i} - 4\,xy \cdot \vec{j}, \qquad |\vec{a}| = \sqrt{10}, \qquad \vec{e_a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{\sqrt{10}}$$

$$\frac{\partial\varphi}{\partial\vec{a}} = \nabla\varphi(x,y) \cdot \vec{e_a} = -\frac{3}{\sqrt{10}}\,x^2 + \sqrt{\frac{2}{5}}\,y^2 - \frac{6\,\sqrt{2}}{\sqrt{5}}\,x\,y,$$

$$\left.\frac{\partial\varphi}{\partial\vec{a}}\right|_{x=1,\,y=-2} = \frac{29}{\sqrt{10}} \simeq 9.17$$

Aufgaben:

a) $\varphi(x,y) = x^2 + xy, \qquad \vec{a} = (2,-1), \qquad P = (1,-2)$

b) $\varphi(x,y) = x^2 + e^{y-x}, \qquad \vec{a} = (1,2), \qquad P = (1,1)$

c) $\varphi(x,y,z) = x^2 + y \cdot z - x \cdot z, \qquad \vec{a} = (1,2,-1), \qquad P = (2,1,-3)$

d) $\varphi(x,y,z) = x\,y\,z - 4\,x\,e^y, \qquad \vec{a} = (3,2,1), \qquad P = (3,0,-1)$

3. Divergenz

A8 Bestimmen Sie die Divergenz folgender Vektorfelder:

a) $\vec{r} = x\,\vec{i} + y\,\vec{j}, \qquad \vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k},$

b) $\vec{F_1} = x\,\vec{i} + 2\,y\,\vec{j} - 3\,z\,\vec{k}, \qquad \vec{F_2} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$

c) $\vec{F_1} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}, \qquad \vec{F_2} = x^2\,\vec{i} - 2\,y^2\,\vec{j} + 4\,z^2\,\vec{k}$

d) $\vec{F_1} = \dfrac{x}{x^2+y^2}\,\vec{i} + \dfrac{y}{x^2+y^2}\,\vec{j}, \qquad \vec{F_2} = \dfrac{xy}{x^2+y^2}\,\vec{i} + \dfrac{xy}{x^2+y^2}\,\vec{j}$

e) $\vec{F_1} = e^{x^2}\,\vec{i} + e^{y^2}\,\vec{j} + e^{z^2}\,\vec{k}, \qquad \vec{F_2} = y\,e^{-x}\,\vec{i} + z\,e^{3y}\,\vec{j} + x\,e^{2z}\,\vec{k}$

f) $\vec{F_1} = y^2\,e^{-x} + xz\,e^{-y} + xy\,e^{-z}, \qquad \vec{F_2} = x\,e^{-y} + y\,e^{-z} + z\,e^{-x}$

A9 Bestimmen Sie die Divergenz folgender Vektorfelder im Punkt P:

a) $\vec{F} = xyz\,\vec{i} + xz\,\vec{j} + z\,\vec{k}$, $\quad P_1 = (3,\ 2,\ -0.5)$, $\quad P_2 = (7,\ 1,\ -1)$, $\quad P_3 = (7,\ 3,\ -2)$

b) $\vec{F} = x^3\,\vec{i} + y^2\,\vec{j} + z\,\vec{k}$, $\quad P_1 = (1,\ -2,\ 4)$, $\quad P_2 = (-2,\ -1,\ 12)$

c) $\vec{F} = x\,e^x\,\vec{i} + y\,e^y\,\vec{j} + z\,e^z\,\vec{k}$, $\quad P_1 = (0,\ 0,\ 0)$, $\quad P_2 = (1,\ 0,\ 0)$, $\quad P_3 = (0,\ 2,\ 0)$

d) $\vec{F} = y\,e^x\,\vec{i} + z\,e^y\,\vec{j} + x\,e^z\,\vec{k}$, $\quad P_1 = (1,\ 1,\ 1)$, $\quad P_2 = (2,\ 1,\ 0)$

e) $\vec{F} = e^x\,\vec{r} = e^x\,(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})$, $\quad P_1 = \left(0,\ 3,\ \frac{1}{3}\right)$, $\quad P_2 = \left(1,\ -5,\ -\frac{1}{7}\right)$

f) $\vec{F} = e^{x^2}\,\vec{r} = e^{x^2}\,(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})$, $\quad P_1 = (0,\ 1,\ 1)$, $\quad P_2 = (2,\ -3,\ 4)$

g) $\vec{F} = e^{r^2}\,(\vec{i} + \vec{j} + \vec{k})$, $\quad P_1 = (1,\ -1,\ 0)$, $\quad P_2 = (1,\ 0,\ 0)$, $\quad P_3 = (1,\ 1,\ -1)$

h) $\vec{F} = x\,e^{y^2}\,\vec{i} + x\,e^{z^2}\,\vec{j} + xz\,e^{x^2}\,\vec{k}$, $\quad P_1 = (1,\ 0,\ -1)$, $\quad P_2 = (0,\ 0,\ 3)$

4. Rotation

Bestimmen Sie die Rotation folgender Vektorfelder

A10

a) $\vec{F} = \ln x \cdot \vec{i} + \ln(xy) \cdot \vec{j}$

b) $\vec{F} = \ln y \cdot \vec{i} + \ln(xy) \cdot \vec{j} + \ln z \cdot \vec{k}$

c) $\vec{F} = \ln y \cdot \vec{i} + \ln z \cdot \vec{j} + \ln x \cdot \vec{k}$,

d) $\vec{F} = \ln(xy) \cdot \vec{i} + \ln(yz) \cdot \vec{j} + \ln(xz) \cdot \vec{k}$

e) $\vec{F} = \ln(xyz) \cdot \vec{i} + \ln y \cdot \vec{j} + \ln y \cdot \vec{k}$

A11

a) $\vec{F} = e^y \cdot \vec{i} + e^z \cdot \vec{j} + e^x \cdot \vec{k}$, $\quad$ b) $\vec{F} = e^{y^2} \cdot \vec{i} + e^{z^2} \cdot \vec{j} + e^{x^2} \cdot \vec{k}$

c) $\vec{F} = \dfrac{e^y}{x} \cdot \vec{i} + \dfrac{e^z}{y} \cdot \vec{j} + \dfrac{e^x}{z} \cdot \vec{k}$, $\quad$ d) $\vec{F} = e^{xy} \cdot \vec{i} + e^{yz} \cdot \vec{j} + e^{xz} \cdot \vec{k}$,

e) $\vec{F} = e^{x+y} \cdot \vec{i} + e^{y+z} \cdot \vec{j} + e^{x+z} \cdot \vec{k}$, $\quad$ f) $\vec{F} = e^x \cdot \vec{i} + e^y \cdot \vec{j} + e^z \cdot \vec{k}$

A12

a) $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$, $\quad$ b) $\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$

c) $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$, $\quad$ d) $\vec{F} = \dfrac{x\,\vec{i} + y\,\vec{j} + z\,\vec{k}}{x^2 + y^2 + z^2}$

e) $\vec{F} = y\,e^x\,\vec{i} + z\,e^y\,\vec{j} + x\,e^z\,\vec{k}$, $\quad$ f) $\vec{F} = y^2\,e^{-x}\,\vec{i} + xz\,e^{-y}\,\vec{j} + xy\,e^{-z}\,\vec{k}$

5. Laplace-Operator

A13 Wenden Sie den Laplace-Operator auf folgende Funktionen an:

$$a)\quad f(x,y,z) = x^2 + y^2 + z^2$$
$$b)\quad f(x,y,z) = x^2\, y^2\, z$$
$$c)\quad f(x,y,z) = x^2\, y^3\, z^4$$
$$d)\quad f(\vec{r}) = \cos\, r$$
$$e)\quad f(\vec{r}) = e^r$$

6. Linienintegrale

A14 Bestimmen Sie die Arbeit, die das ebene Kraftfeld $\vec{F}(x,y)$ auf einen Massenpunkt bei einer Verschiebung längs eines kreislinigen Segments vom Punkt $A\,(1,0)$ nach Punkt $B\,(0,1)$ verrichtet

$$a)\quad \vec{F} = (0,\ x^2), \qquad\qquad b)\quad \vec{F} = (y^2,\ 0)$$

A15 Bestimmen Sie die Arbeit, die das ebene Kraftfeld $\vec{F}(x,y)$ auf einen Massenpunkt bei einer geradlinigen Verschiebung von A nach B verrichtet

$$a)\quad \vec{F} = (x^2 y,\ \sin(\pi y)), \qquad A = (0,2), \quad B = (1,4)$$
$$b)\quad \vec{F} = (x^2 y,\ \sin(\pi y)), \qquad A = (0,2), \quad B = (2,4)$$

A16 Welchen Wert besitzt das Linienintegral des räumlichen Vektorfeldes $\vec{F}(x,y,z)$ längs der Kurve C, die durch den Ortsvektor $\vec{r}(t)$ beschrieben wird

$$a)\quad \vec{F} = (xy,\ yz,\ xz), \qquad \vec{r}(t) = (t,\ t^2,\ t^3), \qquad t \in [-1,\ 1]$$
$$b)\quad \vec{F} = (x,\ y,\ z), \qquad \vec{r}(t) = (t,\ t^2,\ t^3), \qquad t \in [0,\ 1]$$
$$c)\quad \vec{F} = (2x + y^2,\ y + 3,\ x + z), \qquad \vec{r}(t) = (t,\ t^2,\ t), \qquad t \in [0,\ 2]$$

A17 Bestimmen Sie den Wert des Linienintegrals

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$a)\quad \vec{F} = (x,\ y,\ z), \qquad \vec{r}(t) = (\sin t,\ \cos t,\ t^2), \qquad t \in \left[0,\ \frac{\pi}{2}\right]$$
$$b)\quad \vec{F} = \vec{F} = (x + y,\ y,\ z), \qquad \vec{r}(t) = (\sin t,\ \cos t,\ t^2), \qquad t \in \left[0,\ \frac{\pi}{2}\right]$$
$$c)\quad \vec{F} = (x + y,\ y + z,\ z), \qquad \vec{r}(t) = (\sin t,\ \cos t,\ t), \qquad t \in [0,\ \pi]$$
$$d)\quad \vec{F} = (x,\ yz,\ z^2 - x), \qquad \vec{r}(t) = (t^2,\ 1 - t,\ t), \qquad t \in [0,\ 1]$$

A18 Bestimmen Sie den Wert des Linienintegrals

$$\int_{AB} \left[(x^2 - 2xy)\, dx + (2xy + y^2)\, dy \right]$$

wo AB ist Kurvensegment der Funktion $y = x^2$ zwischen den Punkten $A(1,\ 1)$ und $B(2,\ 4)$.

A19 Bestimmen Sie folgendes Linienintegral

$$\int_{C} \left(y^2\, dx + x^2\, dy \right)$$

wo C ist die obere Hälfte der Ellipse $x = a \cos\, t,\ \ y = b\, \sin t,$ Richtung im Uhrzeigersinn.

7. Ebene und räumlich Kurven: Lösungen

7.1. Differentiation eines Vektors nach einem Parameter

L1

$$a)\quad \vec{r}(t) = \begin{pmatrix} t+t^2 \\ t^3 \\ \sqrt{t} \end{pmatrix}, \qquad \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} 1+2t \\ 3\,t^2 \\ \frac{1}{2\sqrt{t}} \end{pmatrix}, \qquad \frac{d^2\vec{r}(t)}{dt^2} = \begin{pmatrix} 2 \\ 6\,t \\ -\frac{1}{4t\sqrt{t}} \end{pmatrix},$$

$$b)\quad \vec{r}(t) = \begin{pmatrix} t\cos t \\ t^2\sin t \\ \sqrt[3]{t} \end{pmatrix}, \qquad \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} \cos t - t\,\sin t \\ 2t\,\sin t + t^2\,\cos t \\ \frac{1}{3\,\sqrt[3]{t^2}} \end{pmatrix} \qquad \frac{d^2\vec{r}(t)}{dt^2} = \begin{pmatrix} -2\sin t - t\,\cos t \\ (2-t^2)\,\sin t + 4t\,\cos t \\ -\frac{2}{9\,t\,\sqrt[3]{t^2}} \end{pmatrix}$$

$$c)\quad \vec{r}(t) = \begin{pmatrix} e^{2t} \\ e^{t^2} \\ t \end{pmatrix}, \qquad \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} 2\,e^{2t} \\ 2\,t\,e^{t^2} \\ 1 \end{pmatrix}, \qquad \frac{d^2\vec{r}(t)}{dt^2} = \begin{pmatrix} 4\,e^{2t} \\ 2\,(1+2\,t^2)\,e^{t^2} \\ 0 \end{pmatrix}$$

8. Gradient, Richtungsableitung

8.1. Gradient eines Skalarfeldes

L2

$$a)\quad \varphi(x,y) = 2x^2 - xy, \qquad \nabla\varphi(x,y) = (4x-y)\,\vec{i} - x\,\vec{j}$$

$$b)\quad \varphi(x,y) = \frac{x}{y}, \qquad \nabla\varphi(x,y) = \frac{\vec{i}}{y} - \frac{x\,\vec{j}}{y^2}$$

$$c)\quad \varphi(x,y) = \frac{y}{x^2+y^2}, \qquad \nabla\varphi(x,y) = \frac{1}{(x^2+y^2)^2}\left(-2xy\,\vec{i} + (x^2-y^2)\,\vec{j}\right)$$

$$d)\quad \varphi(x,y) = \sqrt{x^2+y^2}, \qquad \nabla\varphi(x,y) = \frac{1}{\sqrt{x^2+y^2}}\left(x\,\vec{i} + y\,\vec{j}\right)$$

$$e)\quad \varphi(x,y,z) = \sqrt{x^2+y^2+z^2}, \qquad \nabla\varphi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)$$

L3

$$a)\quad \varphi(x,y) = x\,e^y - y\,e^x, \qquad \nabla\varphi(x,y) = (e^y - y\,e^x)\,\vec{i} + (x\,e^y - e^x)\,\vec{j}$$

$$b)\quad \varphi(x,y) = x\,e^{2y}, \qquad \nabla\varphi(x,y) = e^{2y}\,(\vec{i} + 2x\,\vec{j})$$

$$c)\quad \varphi(x,y) = y\,e^{x^2}, \qquad \nabla\varphi(x,y) = e^{x^2}\,(2xy\,\vec{i} + \vec{j})$$

$$d)\quad \varphi(x,y) = e^{x-y} + e^{y-x}, \qquad \nabla\varphi(x,y) = (e^{x-y} - e^{y-x})\,(\vec{i} - \vec{j})$$

$$e)\quad \varphi(x,y,z) = e^{xyz}, \qquad \nabla\varphi(x,y,z) = e^{xyz}\left(yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}\right)$$

$$f)\quad \varphi(x,y,z) = xy^2\,e^z, \qquad \nabla\varphi(x,y,z) = y\,e^z\left(y\,\vec{i} + 2x\,\vec{j} + xy\,\vec{k}\right)$$

$$g)\quad \varphi(x,y,z) = x\,e^{y-z}, \qquad \nabla\varphi(x,y,z) = e^{y-z}\left(\vec{i} + x\,\vec{j} - x\,\vec{k}\right)$$

$$h)\quad \varphi(x,y,z) = x\,e^{z^2-y}, \qquad \nabla\varphi(x,y,z) = e^{z^2-y}\left(\vec{i} - x\,\vec{j} + 2xz\,\vec{k}\right)$$

L4

$$a)\quad \varphi(x,y) = \sin\, x + \cos(2y), \qquad \nabla\varphi(x,y) = \cos\, x\,\vec{i} - 2\sin\,(2y)\,\vec{j}$$

$$b)\quad \varphi(x,y) = x\,\sin(y^2-3), \qquad \nabla\varphi(x,y) = \sin(y^2-3)\,\vec{i} + 2xy\,\cos(y^2-3)\,\vec{j}$$

$$c)\quad \varphi(x,y) = \cos\left(\frac{x}{y}\right), \qquad \nabla\varphi(x,y) = \sin\left(\frac{x}{y}\right)\left(-\frac{\vec{i}}{y} + \frac{x\,\vec{j}}{y^2}\right)$$

$$d)\quad \varphi(x,y,z) = \sin\,(xyz), \qquad \nabla\varphi(x,y,z) = \cos\,(xyz)\,\left(yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}\right)$$

$$e)\quad \varphi(x,y,z) = \cos\left(\frac{xy}{z}\right) \qquad \nabla\varphi(x,y,z) = \sin\left(\frac{xy}{z}\right)\left(-\frac{y}{z}\,\vec{i} - \frac{x}{z}\,\vec{j} + \frac{xy}{z^2}\,\vec{k}\right)$$

$$f)\quad \varphi(x,y,z) = x\,\ln(y+\cos\, z), \qquad \nabla\varphi(x,y,z) = \ln(y+\cos\, z)\,\vec{i} + \frac{x\,\vec{j}}{y+\cos\, z} - \frac{x\,\sin z\,\vec{k}}{y+\cos\, z}$$

L5

$$a)\quad \varphi(x,y) = x\,\ln(2x-y), \qquad \nabla\varphi(x,y) = \left(\ln(2x-y) + \frac{2x}{2x-y}\right)\vec{i} - \frac{x\vec{j}}{2x-y}$$

$$b)\quad \varphi(x,y,z) = x\,\ln\, y - z\,\ln\, x, \qquad \nabla\varphi(x,y,z) = \left(\ln\, y - \frac{z}{x}\right)\vec{i} + \frac{x}{y}\,\vec{j} - \ln\, x\,\vec{k}$$

$$c)\quad \varphi(x,y,z) = \ln(x+y+z), \qquad \nabla\varphi(x,y,z) = \frac{\vec{i}+\vec{j}+\vec{k}}{x+y+z}$$

L6

$$a)\quad \varphi(x,y) = x^2 + y^2, \qquad P = (1,\ 1)$$

$$\nabla\varphi(x,y) = 2x\vec{i} + 2y\vec{j}, \qquad \nabla\varphi(x,y)\Big|_{x=1,y=1} = 2\vec{i} + 2\vec{j}, \qquad |\nabla\varphi(x,y)| = 2\,\sqrt{2} \simeq 2.83$$

$$b)\quad \varphi(x,y) = x^2 + y\,e^{x-1}, \qquad P = (1,\ 2)$$

$$\nabla\varphi(x,y) = (2x + y\,e^{x-1})\,\vec{i} + e^{x-1}\,\vec{j}, \qquad \nabla\varphi(x,y)\Big|_{x=1,y=2} = 4\vec{i} + \vec{j}, \qquad |\nabla\varphi(x,y)| = \sqrt{17}$$

$$c)\quad \varphi(x,y,z) = xz + e^y - \sqrt{z}, \qquad P = (-1,\ 0,\ 4)$$

$$\nabla\varphi(x,y,z) = z\,\vec{i} + e^y\,\vec{j} + \left(x - \frac{1}{2\,\sqrt{z}}\right)\vec{k}, \qquad \nabla\varphi(x,y,z)\Big|_{x=-1,y=0,z=4} = 4\,\vec{i} + \vec{j} - \frac{5}{4}\,\vec{k}$$

$$|\nabla\varphi(x,y,z)| = \frac{3}{4}\,\sqrt{33} \simeq 4.31$$

$$d)\quad \varphi(x,y,z) = e^{x-y} + e^{x-z} + e^{y-z}, \qquad P = (2,\ 2,\ 2), \qquad P = (1,\ 1,\ 1)$$

$$\nabla\varphi(x,y,z) = (e^{x-y} + e^{x-z})\,\vec{i} + (e^{y-z} - e^{x-y})\,\vec{j} - (e^{x-z} + e^{y-z})\,\vec{k}$$

$$\nabla\varphi(x,y,z)\Big|_{x=2,y=2,z=2} = 2\,\vec{i} - 2\,\vec{k}, \qquad |\nabla\varphi(x,y,z)| = 2\,\sqrt{2} \simeq 2.83$$

$$\nabla\varphi(x,y,z)\Big|_{x=1,y=1,z=1} = 2\,\vec{i} - 2\,\vec{k}, \qquad |\nabla\varphi(x,y,z)| = 2\,\sqrt{2} \simeq 2.83$$

8.2. Richtungsableitung

L7

a) $\varphi(x,y) = x^2 + xy, \qquad \vec{a} = (2,-1), \qquad P = (1,-2)$

$$\frac{\partial\varphi}{\partial\vec{a}} = \nabla\varphi(x,y)\cdot\vec{e_a} = \frac{2}{\sqrt{5}}(2x+y) - \frac{x}{\sqrt{5}}, \qquad \left.\frac{\partial\varphi}{\partial\vec{a}}\right|_{x=1,\,y=-2} = -\frac{1}{\sqrt{5}} \simeq -0.45$$

b) $\varphi(x,y) = x^2 + e^{y-x}, \qquad \vec{a} = (1,2), \qquad P = (1,1)$

$$\frac{\partial\varphi}{\partial\vec{a}} = \nabla\varphi(x,y)\cdot\vec{e_a} = \frac{1}{\sqrt{5}}\left(2x + e^{y-x}\right), \qquad \left.\frac{\partial\varphi}{\partial\vec{a}}\right|_{x=1,\,y=1} = \frac{3}{\sqrt{5}} \simeq 1.34$$

c) $\varphi(x,y,z) = x^2 + y\cdot z - x\cdot z, \qquad \vec{a} = (1,2,-1), \qquad P = (2,1,-3)$

$$\frac{\partial\varphi}{\partial\vec{a}} = \nabla\varphi(x,y)\cdot\vec{e_a} = \frac{1}{\sqrt{6}}(3x-y+z), \qquad \left.\frac{\partial\varphi}{\partial\vec{a}}\right|_{x=2,\,y=1,\,z=-3} = \sqrt{\frac{2}{3}} \simeq 0.82$$

d) $\varphi(x,y,z) = xyz - 4\,x\,e^{y}, \qquad \vec{a} = (3,2,1), \qquad P = (3,0,-1)$

$$\frac{\partial\varphi}{\partial\vec{a}} = \nabla\varphi(x,y)\cdot\vec{e_a} = -\frac{1}{\sqrt{14}}\left(-3yz - 2xz - xy + 4(3+2x)e^{y}\right)$$

$$\left.\frac{\partial\varphi}{\partial\vec{a}}\right|_{x=3,\,y=0,\,z=-1} = -3\sqrt{14} \simeq -11.22$$

9. Divergenz

L8

a) $\vec{r} = x\,\vec{i} + y\,\vec{j}, \qquad div\,\vec{r} = 2$

$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}, \qquad div\,\vec{r} = 3$

b) $\vec{F_1} = x\,\vec{i} + 2y\,\vec{j} - 3z\,\vec{k}, \qquad div\,\vec{F_1} = 0$

$\vec{F_2} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}, \qquad div\,\vec{F_2} = 2(x+y+z)$

c) $\vec{F_1} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}, \qquad div\,\vec{F_1} = 0$

$\vec{F_2} = x^2\,\vec{i} - 2y^2\,\vec{j} + 4z^2\,\vec{k}, \qquad div\,\vec{F_2} = 2(x-2y+4z)$

d) $\vec{F_1} = \dfrac{x}{x^2+y^2}\,\vec{i} + \dfrac{y}{x^2+y^2}\,\vec{j}, \qquad div\,\vec{F_1} = 0$

$\vec{F_2} = \dfrac{xy}{x^2+y^2}\,\vec{i} + \dfrac{xy}{x^2+y^2}\,\vec{j}, \qquad div\,\vec{F_2} = \dfrac{(x^2-y^2)\cdot(x-y)}{(x^2+y^2)^2}$

e) $\vec{F_1} = e^{x^2}\,\vec{i} + e^{y^2}\,\vec{j} + e^{z^2}\,\vec{k}, \qquad div\,\vec{F_1} = 2\left(x\,e^{x^2} + y\,e^{y^2} + z\,e^{z^2}\right)$

$\vec{F_2} = y\,e^{-x}\,\vec{i} + z\,e^{3y}\,\vec{j} + x\,e^{2z}\,\vec{k}, \qquad div\,\vec{F_2} = -y\,e^{-x} + 3z\,e^{3y} + 2x\,e^{2z}$

f) $\vec{F_1} = y^2\,e^{-x}\,\vec{i} + xz\,e^{-y}\,\vec{j} + xy\,e^{-z}\,\vec{k}, \qquad div\,\vec{F_1} = -y^2\,e^{-x} - xz\,e^{-y} - xy\,e^{-z}$

$\vec{F_2} = x\,e^{-y}\,\vec{i} + y\,e^{-z}\,\vec{j} + z\,e^{-x}\,\vec{k}, \qquad div\,\vec{F_2} = e^{-y} + e^{-z} + e^{-x}$

L9

a) $\vec{F} = xyz\,\vec{i} + xz\,\vec{j} + z\,\vec{k}, \qquad div\,\vec{F} = 1 + yz,$

$div\,\vec{F}\Big|_{P_1=(3,\,2,\,-0.5)} = 0, \qquad div\,\vec{F}\Big|_{P_2=(7,\,1,\,-1)} = 0, \qquad div\,\vec{F}\Big|_{P_3=(7,\,3,\,-2)} = -5$

b) $\vec{F} = x^3\,\vec{i} + y^2\,\vec{j} + z\,\vec{k}, \qquad div\,\vec{F} = 1 + 3\,x^2 + 2y,$

$div\,\vec{F}\Big|_{P_1=(1,\,-2,\,4)} = 0, \qquad div\,\vec{F}\Big|_{P_2=(-2,\,-1,\,12)} = 11$

c) $\vec{F} = x\,e^x\,\vec{i} + y\,e^y\,\vec{j} + z\,e^z\,\vec{k}, \qquad div\,\vec{F} = (1+x)\,e^x + (1+y)\,e^y + (1+z)\,e^z$

$div\,\vec{F}\Big|_{P_1=(0,\,0,\,0)} = 3, \qquad div\,\vec{F}\Big|_{P_2=(1,\,0,\,0)} = 2\,(e+1), \qquad div\,\vec{F}\Big|_{P_3=(0,\,2,\,0)} = 2 + 3\,e^2$

d) $\vec{F} = y\,e^x\,\vec{i} + z\,e^y\,\vec{j} + x\,e^z\,\vec{k}, \qquad div\,\vec{F} = y\,e^x + z\,e^y + x\,e^z,$

$div\,\vec{F}\Big|_{P_1=(1,\,1,\,1)} = 3\,e, \qquad div\,\vec{F}\Big|_{P_2=(2,\,1,\,0)} = 2 + e^2$

e) $\vec{F} = e^x\,\vec{r} = e^x\,(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}), \qquad div\,\vec{F} = (3+x)\,e^x,$

$div\,\vec{F}\Big|_{P_1=\left(0,\,3,\,\frac{1}{3}\right)} = 3, \qquad div\,\vec{F}\Big|_{P_2=\left(1,\,-5,\,-\frac{1}{7}\right)} = 4\,e$

f) $\vec{F} = e^{x^2}\,\vec{r} = e^{x^2}\,(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}), \qquad div\,\vec{F} = (3 + 2\,x^2)\,e^{x^2},$

$div\,\vec{F}\Big|_{P_1=(0,\,1,\,1)} = 3, \qquad div\,\vec{F}\Big|_{P_2=(2,\,-3,\,4)} = 11\,e^4$

g) $\vec{F} = e^{r^2}\,(\vec{i} + \vec{j} + \vec{k}) = e^{x^2+y^2+z^2}\,(\vec{i} + \vec{j} + \vec{k}), \qquad div\,\vec{F} = 2\,(x+y+z)\,e^{x^2+y^2+z^2},$

$div\,\vec{F}\Big|_{P_1=(1,\,-1,\,0)} = 0, \qquad div\,\vec{F}\Big|_{P_2=(1,\,0,\,0)} = 2\,e, \qquad div\,\vec{F}\Big|_{P_3=(1,\,1,\,-1)} = 2\,e^3$

h) $\vec{F} = x\,e^{y^2}\,\vec{i} + x\,e^{z^2}\,\vec{j} + xz\,e^{x^2}\,\vec{k}, \qquad div\,\vec{F} = e^{y^2} + x\,e^{x^2},$

$div\,\vec{F}\Big|_{P_1=(1,\,0,\,-1)} = 1 + e, \qquad div\,\vec{F}\Big|_{P_2=(0,\,0,\,3)} = 1$

10. Rotation

L10

$$a)\quad \vec{F} = \ln x \cdot \vec{i} + \ln(xy) \cdot \vec{j}, \qquad rot\,\vec{F} = \frac{\vec{k}}{x}$$

$$b)\quad \vec{F} = \ln y \cdot \vec{i} + \ln(xy) \cdot \vec{j} + \ln z \cdot \vec{k}, \qquad rot\,\vec{F} = \left(\frac{1}{x} - \frac{1}{y}\right) \cdot \vec{k}$$

$$c)\quad \vec{F} = \ln y \cdot \vec{i} + \ln z \cdot \vec{j} + \ln x \cdot \vec{k}, \qquad rot\,\vec{F} = -\frac{\vec{i}}{z} - \frac{\vec{j}}{x} - \frac{\vec{k}}{y}$$

$$d)\quad \vec{F} = \ln(xy) \cdot \vec{i} + \ln(yz) \cdot \vec{j} + \ln(xz) \cdot \vec{k}, \qquad rot\,\vec{F} = -\frac{\vec{i}}{z} - \frac{\vec{j}}{x} - \frac{\vec{k}}{y}$$

$$e)\quad \vec{F} = \ln(xyz) \cdot \vec{i} + \ln y \cdot \vec{j} + \ln y \cdot \vec{k}, \qquad rot\,\vec{F} = \frac{\vec{i}}{y} + \frac{\vec{j}}{z} - \frac{\vec{k}}{y}$$

L11

$$a)\quad \vec{F} = e^y \cdot \vec{i} + e^z \cdot \vec{j} + e^x \cdot \vec{k}, \qquad rot\,\vec{F} = -e^z \cdot \vec{i} - e^x \cdot \vec{j} - e^y \cdot \vec{k}$$

$$b)\quad \vec{F} = e^{y^2} \cdot \vec{i} + e^{z^2} \cdot \vec{j} + e^{x^2} \cdot \vec{k}, \qquad rot\,\vec{F} = -2\left(z\,e^{z^2} \cdot \vec{i} + x\,e^{x^2} \cdot \vec{j} + y\,e^{y^2} \cdot \vec{k}\right)$$

$$c)\quad \vec{F} = \frac{e^y}{x} \cdot \vec{i} + \frac{e^z}{y} \cdot \vec{j} + \frac{e^x}{z} \cdot \vec{k}, \qquad rot\,\vec{F} = -\frac{e^z}{y} \cdot \vec{i} - \frac{e^x}{z} \cdot \vec{j} - \frac{e^y}{x} \cdot \vec{k}$$

$$d)\quad \vec{F} = e^{xy} \cdot \vec{i} + e^{yz} \cdot \vec{j} + e^{xz} \cdot \vec{k}, \qquad rot\,\vec{F} = -y\,e^{yz} \cdot \vec{i} - z\,e^{xz} \cdot \vec{j} - x\,e^{xy} \cdot \vec{k}$$

$$e)\quad \vec{F} = e^{x+y} \cdot \vec{i} + e^{y+z} \cdot \vec{j} + e^{x+z} \cdot \vec{k}, \qquad rot\,\vec{F} = -e^{y+z} \cdot \vec{i} - e^{x+z} \cdot \vec{j} - e^{x+y} \cdot \vec{k}$$

$$f)\quad \vec{F} = e^x \cdot \vec{i} + e^y \cdot \vec{j} + e^z \cdot \vec{k}, \qquad rot\,\vec{F} = \vec{0}$$

L12

$$a)\quad \vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}, \qquad rot\,\vec{r} = \vec{0},$$

$$b)\quad \vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}, \qquad rot\,\vec{F} = \vec{0}$$

$$c)\quad \vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}, \qquad rot\,\vec{F} = \vec{0}$$

$$d)\quad \vec{F} = \frac{x\,\vec{i} + y\,\vec{j} + z\,\vec{k}}{x^2 + y^2 + z^2}, \qquad rot\,\vec{F} = \vec{0}$$

$$e)\quad \vec{F} = y\,e^x\,\vec{i} + z\,e^y\,\vec{j} + x\,e^z\,\vec{k}, \qquad rot\,\vec{F} = -e^y\,\vec{i} - e^z\,\vec{j} - e^x\,\vec{k}$$

$$f)\quad \vec{F} = y^2\,e^{-x}\,\vec{i} + xz\,e^{-y}\,\vec{j} + xy\,e^{-z}\,\vec{k}$$
$$rot\,\vec{F} = (x\,e^{-z} - x\,e^{-y})\,\vec{i} - y\,e^{-z}\,\vec{j} + (z\,e^{-y} - 2y\,e^{-x})\,\vec{k}$$

11. Laplace-Operator

L13

$$a)\quad f(x,y,z) = x^2 + y^2 + z^2, \qquad \Delta f(x,y,z) = 6$$

$$b)\quad f(x,y,z) = x^2\, y^2\, z, \qquad \Delta f(x,y,z) = 2z\,(x^2 + y^2)$$

$$c)\quad f(x,y,z) = x^2\, y^3\, z^4, \qquad \Delta f(x,y,z) = 2\,y\,z^2\,(y^2\,z^2 + 3\,x^2\,z^2 + 6x^2\,y^2)$$

$$d)\quad f(\vec{r}) = \cos\, r, \qquad r = \sqrt{x^2 + y^2 + z^2}, \qquad \Delta f(\vec{r}) = -\cos\, r - \frac{2\,\sin r}{r}$$

$$e)\quad f(\vec{r}) = e^r, \qquad \Delta f(\vec{r}) = \left(1 + \frac{2}{r}\right) e^r$$

12. Linienintegrale

L14

$$a)\quad \vec{F} = (0,\ x^2), \qquad C: \quad \vec{r} = (\cos t,\ \sin t), \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

$$W = \int\limits_0^{\frac{\pi}{2}} \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt}\, dt = \int\limits_0^{\frac{\pi}{2}} \cos^3\ t\, dt = \frac{1}{4} \int\limits_0^{\frac{\pi}{2}} (\cos(3t) + 3\ \cos t)\ dt = \frac{2}{3}$$

$$\cos^3\ \alpha = \frac{1}{4}\ (\cos(3\alpha) + 3\ \cos\alpha)$$

$$b)\quad \vec{F} = (y^2,\ 0), \qquad C: \quad \vec{r} = (\cos t,\ \sin t), \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

$$W = \int\limits_0^{\frac{\pi}{2}} \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt}\, dt = -\int\limits_0^{\frac{\pi}{2}} \sin^3\ t\, dt = \frac{1}{4} \int\limits_0^{\frac{\pi}{2}} (3\sin t - \sin(3t))\ dt = -\frac{2}{3}$$

L15

$$a)\quad \vec{F} = (x^2 y,\ \sin(\pi y)), \qquad A = (0,2), \quad B = (1,4), \qquad (AB): \quad y = 2x + 2$$

$$\vec{r} = (t,\ 2 + 2t), \qquad 0 \leqslant t \leqslant 1$$

$$W = \int\limits_0^1 \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt}\, dt = \int\limits_0^1 \Big(2\,(t+1)\,t^2 + 2\sin(2\pi(t+1))\Big)\ dt = \left[\frac{t^4}{2} + \frac{2}{3}\,t^3 - \frac{1}{\pi}\ \cos(2\pi t)\right]_0^1 = \frac{7}{6}$$

$$b)\quad \vec{F} = (x^2 y,\ \sin(\pi y)), \qquad A = (0,2), \quad B = (2,4), \qquad (AB): \quad y = x + 2$$

$$\vec{r} = (t,\ 2 + t), \qquad 0 \leqslant t \leqslant 2$$

$$W = \int\limits_0^2 \Big((2+t)\,t^2 + \sin(\pi\,(2+t))\Big)\ dt = \left[\frac{t^4}{4} + \frac{2}{3}\,t^3 - \frac{\cos(\pi t)}{\pi}\right]_0^2 = \frac{28}{3}$$

L16 Welchen Wert besitzt das Linienintegral des räumlichen Vektorfeldes $\vec{F}(x, y, z)$ längs der Kurve C, die durch den Ortsvektor $\vec{r}(t)$ beschrieben wird

a) $\vec{F} = (xy,\ yz,\ xz), \qquad \vec{r}(t) = (t,\ t^2,\ t^3), \qquad t \in [-1,\ 1]$

$$W = \int_{-1}^{1} \left(t^3 + 5\,t^6\right)\,dt = \frac{10}{7}$$

b) $\vec{F} = (x,\ y,\ z), \qquad \vec{r}(t) = (t,\ t^2,\ t^3), \qquad t \in [0,\ 1]$

$$W = \int_{0}^{1} (t + 2\,t^3 + 3\,t^5)\,dt = \frac{3}{2}$$

c) $\vec{F} = (2x + y^2,\ y + 3,\ x + z), \qquad \vec{r}(t) = (t,\ t^2,\ t), \qquad t \in [0,\ 2]$

$$W = \int_{0}^{2} \left(4t + t^4 + 2t\,(t^2 + 3)\right)\,dt = \frac{172}{5}$$

L17 Bestimmen Sie den Wert des Linienintegrals

$$\int_C \vec{F}\,(\vec{r}) \cdot d\vec{r}$$

a) $\vec{F} = (x,\ y,\ z), \qquad \vec{r}\,(t) = (\sin t,\ \cos t,\ t^2), \qquad t \in \left[0,\ \frac{\pi}{2}\right]$

$$\int_C \vec{F}\,(\vec{r}) \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \vec{F}\,(\vec{r}) \cdot \frac{d\vec{r}}{dt}\,dt = 2\int_0^{\frac{\pi}{2}} t^3\,dt = \frac{\pi^4}{32}$$

b) $\vec{F} = (x + y,\ y,\ z), \qquad \vec{r}\,(t) = (\sin t,\ \cos t,\ t^2), \qquad t \in \left[0,\ \frac{\pi}{2}\right]$

$$\int_C \vec{F}\,(\vec{r}) \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \left(\cos^2 t + 2\,t^3\right)\,dt = \frac{\pi}{4} + \frac{\pi^4}{32} \simeq 3.83$$

c) $\vec{F} = (x + y,\ y + z,\ z), \qquad \vec{r}\,(t) = (\sin t,\ \cos t,\ t), \qquad t \in [0,\ \pi]$

$$\int_C \vec{F}\,(\vec{r}) \cdot d\vec{r} = \int_0^{\pi} (\cos^2 t - \sin\ t + t)\,dt = -\frac{\pi}{2} + \frac{\pi^2}{2} \simeq 3.36$$

d) $\vec{F} = (x,\ yz,\ z^2 - x), \qquad \vec{r}\,(t) = (t^2,\ 1 - t,\ t), \qquad t \in [0,\ 1]$

$$\int_C \vec{F}\,(\vec{r}) \cdot d\vec{r} = \int_0^{1} (2\,t^3 + t^2 - t)\,dt = \frac{1}{3}$$

L18

$$\int_{AB} \left[(x^2 - 2xy)\,dx + (2xy + y^2)\,dy\right] = \int_1^2 \left(t^2 - 2\,t^3 + 4\,t^4 + 2\,t^5\right)\,dt = \frac{1219}{30} \simeq 40.63$$

$C: \quad y = x^2, \quad x(t) = t, \quad y(t) = t^2, \qquad dx = dt, \quad dy = 2\,t\,dt, \qquad 1 \leqslant t \leqslant 2$

$\vec{F} = \left(x^2 - 2xy,\ 2xy + y^2\right) = \left(t^2 - 2\,t^3,\ 2\,t^3 + t^4\right)$

L19

$$\int_C \left(y^2\,dx + x^2\,dy\right) = \int_\pi^0 \left[b^2 \sin^2 t \cdot (-a \sin t) + a^2 \cos^2 t \cdot b \cos t\right] dt =$$

$$= -a\,b^2 \int_\pi^0 \sin^3 t\,dt + a^2\,b \int_\pi^0 \cos^3 t\,dt = -\frac{ab^2}{4} \int_\pi^0 (3 \sin t - \sin(3t))\,dt +$$

$$+ \frac{a^2\,b}{4} \int_\pi^0 (\cos(3t) + 3 \cos t)\,dt = \frac{4}{3}\,ab^2$$

$x = a \cos t, \quad y = b \sin t, \qquad dx = -a \sin t\,dt, \quad dy = b \cos t\,dt$