Funktionen mehrerer Variablen: Differentialrechnung

Aufgaben

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1. Partielle Ableitungen 1. Ordnung

Bestimmen Sie die partiellen Ableitungen 1. Ordnung folgender Funktionen:

A1

a) $f(x, y) = x \cos(xy)$ $\quad g(x, y) = \cos(x + y) + \sin(xy)\,,$ $\quad h(x, y) = x^2 \sin(x + y)$

b) $f(x, y) = \ln(x + y)\,,$ $\quad g(x, y) = \ln(xy) + \sqrt{x^2 - y}\,,$ $\quad h(x, y) = \ln(y^2 - x) + \cos x$

c) $f(x, y) = e^x \sin y\,,$ $\quad g(x, y) = e^{x+y}\,,$ $\quad h(x, y) = e^{x^2+y^2}$

d) $f(x, y) = \sin(\sqrt{x} + y)$ $\quad g(x, y) = \sin(\sqrt{x + 2y})$

e) $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y^2}$ $\quad g(x, y) = \dfrac{1}{xy^2} + x^2 y$ $\quad h(x, y) = \dfrac{1}{x^2 + y^2}$

2. Linearisierung einer Funktion, Gleichung der Tangentialebene

Die lineare Näherung einer im Punkt $P(x_0, y_0)$ differenzierbaren Funktion $f(x, y)$ ist eine Funktion

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \tag{1}$$

Die lineare Näherung einer im Punkt $P(x_0, y_0, z_0)$ differenzierbaren Funktion $f(x, y, z)$ ist eine Funktion

$$\begin{aligned} L(x, y, z) &= f(x_0, y_0, z_0) + \frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) \\ &+ \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) \end{aligned} \tag{2}$$

Beispiel 1: Wir linearisieren die Funktion $f(x,y) = (x^2 - y^2) \cdot \sin y$ im Punkt $P(1, \pi/2)$

$$f(x,y) = (x^2 - y^2) \cdot \sin y, \qquad P\left(1, \frac{\pi}{2}\right), \qquad x_0 = 1, \quad y_0 = \frac{\pi}{2}$$

$$\frac{\partial f(x,y)}{\partial x} = 2x \sin y, \qquad \left[\frac{\partial f(x,y)}{\partial x}\right]_{x=1,\, y=\pi/2} = 2 \sin\left(\frac{\pi}{2}\right) = 2$$

$$\frac{\partial f(x,y)}{\partial y} = -2y \sin y + (x^2 - y^2) \cos y, \qquad \left[\frac{\partial f(x,y)}{\partial x}\right]_{x=1,\, y=\pi/2} = -\pi$$

$$f(x_0, y_0) = f\left(1, \frac{\pi}{2}\right) = 1 - \frac{\pi^2}{4},$$

$$L(x, y) = 2x - \pi y - 1 + \frac{\pi^2}{4} \simeq 2x - 3.14y + 1.47$$

Beispiel 2: Wir linearisieren die Funktion $f(x,y) = x^3 + 2\cos y$ im Punkt $P(-2, 3\pi/2)$

$$f(x,y) = x^3 + 2\cos y, \qquad P\left(-2, \frac{3\pi}{2}\right), \qquad x_0 = -2, \quad y_0 = \frac{3\pi}{2}$$

$$\frac{\partial f(x,y)}{\partial x} = 3x^2, \qquad \left[\frac{\partial f(x,y)}{\partial x}\right]_{x=-2,\, y=3\pi/2} = 12$$

$$\frac{\partial f(x,y)}{\partial y} = -2\sin y, \qquad \left[\frac{\partial f(x,y)}{\partial x}\right]_{x=-2,\, y=3\pi/2} = 2$$

$$f(x_0, y_0) = x_0^3 + 2\cos y_0 = -8$$

$$L(x, y) = 12x + 2y + 16 - 3\pi \simeq 12x + 2y + 6.58$$

Aufgaben: Linearisieren Sie die Funktion $f(x,y)$ im Punkt $P(x_0, y_0)$:

A2

a) $f(x,y) = x^2 + 2xy - 6$, 1) $P(1, 1)$, 2) $P(2, -3)$, 3) $P(0, 0)$

b) $f(x,y) = x^3 + 2y^2 + x$, 1) $P(1, -1)$, 2) $P(0, -2)$

c) $f(x,y) = (x + 3y - 2)^2$, 1) $P(2, -1)$, 2) $P(1, -2)$

d) $f(x,y) = e^{2x-y}$, 1) $P(1, 2)$, 2) $P(1, 1)$

e) $f(x,y) = \ln x + \ln y$, 1) $P(1, 1)$, 2) $P(e, 2e)$

f) $f(x,y) = \sqrt{x} + \sqrt{y}$, 1) $P(1, 4)$, 2) $P(4, 9)$

A3

a) $f(x,y) = 3\sin x + y^3$, 1) $P\left(\frac{\pi}{2}, 2\right)$, 2) $P\left(\frac{\pi}{3}, -1\right)$, 3) $P(0, 1)$

b) $f(x,y) = \sin(x + y)$, 1) $P\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, 2) $P\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, 3) $P\left(\frac{\pi}{8}, \frac{\pi}{8}\right)$

c) $f(x,y) = e^x \cos y$, 1) $P(0, 0)$, 2) $P\left(0, \frac{\pi}{3}\right)$

2.1. Das totale Differential

Unter dem *totalen Differential* einer Funktion von zwei oder drei unabhängigen Veränderlichen versteht man lineare Differentialausdrücke

$$df(x, y) = f_x\, dx + f_y\, dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{3}$$

$$df(x, y, z) = f_x\, dx + f_y\, dy + f_z\, dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

A4

a) $f(x,y) = x^2 + y^2 - 4xy, \qquad g(x,y) = x^3 y^2$

b) $f(x,y) = \dfrac{x}{x^2 - y^2}, \qquad g(x,y) = \dfrac{1}{xy}$

c) $f(x,y) = \sin^2 x + \cos y, \qquad g(x,y) = x^2 + \sin(2x + y)$

d) $f(x,y) = xy - \tan y, \qquad g(x,y) = \sin(x + y) + \cos(x - y)$

e) $f(x,y) = e^{2x-3y}, \qquad g(x,y) = e^{x^2+y^2}$

f) $f(x,y) = \ln(x + y), \qquad g(x,y) = \ln(x^2 + y^2), \qquad h(x,y) = \ln\left(1 + \dfrac{y}{x}\right)$

g) $f(x,y) = \sqrt{x^2 + y^2}, \qquad g(x,y) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}}$

3. Partielle Ableitungen 1. Ordnung: Lösungen

L1

a) $f(x, y) = x \cos(xy), \quad \frac{\partial f}{\partial x} = \cos(xy) - xy \sin(xy), \quad \frac{\partial f}{\partial y} = -x^2 \sin(xy)$

$g(x, y) = \cos(x + y) + \sin(xy)$

$\frac{\partial g}{\partial x} = -\sin(x + y) + y \cos(xy), \quad \frac{\partial g}{\partial y} = -\sin(x + y) + x \cos(xy)$

$h(x, y) = x^2 \sin(x + y)$

$\frac{\partial h}{\partial x} = 2x \sin(x + y) + x^2 \cos(x + y), \quad \frac{\partial h}{\partial y} = x^2 \cos(x + y)$

b) $f(x, y) = \ln(x + y), \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{1}{x + y}$

$g(x, y) = \ln(xy) + \sqrt{x^2 - y}, \quad \frac{\partial g}{\partial x} = \frac{1}{x} + \frac{x}{\sqrt{x^2 - y}}, \quad \frac{\partial g}{\partial y} = \frac{1}{y} - \frac{1}{2\sqrt{x^2 - y}}$

$h(x, y) = \ln(y^2 - x) + \cos x, \quad \frac{\partial h}{\partial x} = -\frac{1}{y^2 - x} - \sin x, \quad \frac{\partial h}{\partial y} = \frac{2y}{y^2 - x}$

c) $f(x, y) = e^x \sin y, \quad \frac{\partial f}{\partial x} = e^x \sin y, \quad \frac{\partial g}{\partial y} = e^x \cos y$

$g(x, y) = e^{x+y}, \quad \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = e^{x+y}$

$h(x, y) = e^{x^2+y^2}, \quad \frac{\partial h}{\partial x} = 2x\, e^{x^2+y^2}, \quad \frac{\partial h}{\partial y} = 2y\, e^{x^2+y^2}$

d) $f(x, y) = \sin(\sqrt{x} + y), \quad \frac{\partial f}{\partial x} = \frac{1}{2} \frac{\cos(\sqrt{x} + y)}{\sqrt{x}}, \quad \frac{\partial f}{\partial y} = \cos(\sqrt{x} + y)$

$g(x, y) = \sin(\sqrt{x + 2y}), \quad \frac{\partial g}{\partial x} = \frac{1}{2} \frac{\cos(\sqrt{x + 2y})}{\sqrt{x + 2y}}, \quad \frac{\partial g}{\partial y} = \frac{\cos(\sqrt{x + 2y})}{\sqrt{x + 2y}}$

e) $f(x, y) = \frac{1}{x} + \frac{1}{y^2}, \quad \frac{\partial f}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial f}{\partial y} = -\frac{2}{y^3}$ (4)

$g(x, y) = \frac{1}{x y^2} + x^2 y, \quad \frac{\partial g}{\partial x} = -\frac{1}{x^2 y^2} + 2xy, \quad \frac{\partial g}{\partial y} = -\frac{2}{x y^3} + x^2$

$h(x, y) = \frac{1}{x^2 + y^2}, \quad \frac{\partial h}{\partial x} = -\frac{2x}{(x^2 + y^2)^2}, \quad \frac{\partial h}{\partial y} = -\frac{2y}{(x^2 + y^2)^2}$

4. Linearisierung einer Funktion, Gleichung der Tangentialebene: Lösungen

L2

a) $f(x,y) = x^2 + 2xy - 6,$

1) $P(1,\ 1):\quad L(x,y) = 4x + 2y - 9,$

2) $P(2,\ -3):\quad L(x,y) = -2x + 4y + 2,$

3) $P(0,\ 0):\quad L(x,y) = 6$

b) $f(x,y) = x^3 + 2y^2 + x,$

1) $P(1,\ -1):\quad L(x,y) = 4x - 4y - 4,$

2) $P(0,\ -2):\quad L(x,y) = x - 8y - 8$

c) $f(x,y) = (x + 3y - 2)^2,$

1) $P(2,\ -1):\quad L(x,y) = -6x - 18y + 3,$

2) $P(1,\ -2):\quad L(x,y) = -14x - 42y - 21$

d) $f(x,y) = e^{2x-y},$

1) $P(1,\ 2):\quad L(x,y) = 2x - y + 1,$

2) $P(1,\ 1):\quad L(x,y) = e(2x - y)$

e) $f(x,y) = \ln x + \ln y,$

1) $P(1,\ 1):\quad L(x,y) = 2x - y + 1, x + y - 2,$

2) $P(e,\ 2e):\quad L(x,y) = \frac{1}{e}\left(x + \frac{y}{2}\right) + \ln 2$

f) $f(x,y) = \sqrt{x} + \sqrt{y},$

1) $P(1,\ 4):\quad L(x,y) = \frac{x}{2} + \frac{y}{4} + \frac{3}{2},$

2) $P(4,\ 9):\quad L(x,y) = \frac{x}{4} + \frac{y}{6} + \frac{5}{2}$

L3

a) $f(x,y) = 3\sin x + y^3, \qquad 1)\ P\left(\frac{\pi}{2},\ 2\right), \qquad 2)\ P\left(\frac{\pi}{3},\ -1\right), \qquad 3)\ P(0,\ 1)$

$$\frac{\partial f(x,y)}{\partial x} = 3\cos x, \qquad \frac{\partial f(x,y)}{\partial y} = 3y^2,$$

a) $P\left(\frac{\pi}{2},\ 2\right): \quad z = 12y - 13$

b) $P\left(\frac{\pi}{3},\ -1\right): \quad z = \frac{3}{2}x + 3y + 2 - \frac{\pi}{2} + \frac{3\sqrt{3}}{2} \simeq 1.5x + 3y + 3.03$

c) $P(0,\ 1): \quad z = 3x + 3y - 2$

b) $f(x,y) = \sin(x+y),$

1) $P\left(\frac{\pi}{6},\ \frac{\pi}{3}\right): \quad L(x,y) = 1,$

2) $P\left(\frac{\pi}{4},\ \frac{\pi}{4}\right): \quad L(x,y) = 1,$

3) $P\left(\frac{\pi}{8},\ \frac{\pi}{8}\right): \quad L(x,y) = \frac{1}{\sqrt{2}}\left(x + y + 1 - \frac{\pi}{4}\right) \simeq 0.707x + 0.707y + 0.152$

c) $f(x,y) = e^x \cos y,$

1) $P(0,\ 0): \quad L(x,y) = x + 1,$

2) $P\left(0,\ \frac{\pi}{3}\right): \quad L(x,y) = \frac{x}{2} - \frac{\sqrt{3}}{2}y + \frac{\pi}{2\sqrt{3}} + \frac{1}{2} \simeq 0.5x - 0.87y + 1.41$

L4

a) $f(x,y) = x^2 + y^2 - 4xy, \qquad df = 2(x-2y)\,dx + 2(y-2x)\,dy$

$g(x,y) = x^3 y^2, \qquad dg = 3x^2y^2\,dx + 2x^3y\,dy$

b) $f(x,y) = \dfrac{x}{x^2-y^2}, \qquad df = -\dfrac{x^2+y^2}{(x^2-y^2)^2}\,dx + \dfrac{2xy}{(x^2-y^2)^2}\,dy$

$g(x,y) = \dfrac{1}{xy}, \qquad dg = -\dfrac{dx}{x^2y} - \dfrac{dy}{xy^2} = -\dfrac{y\,dx + x\,dy}{x^2y^2}$

c) $f(x,y) = \sin^2 x + \cos y, \qquad df = 2\sin x \cdot \cos x\,dx - \sin y\,dy$

$g(x,y) = x^2 + \sin(2x+y), \qquad dg = 2(x + \cos(2x+y))\,dx + \cos(2x+y)\,dy$

d) $f(x,y) = xy - \tan y, \qquad df = y\,dx + (x - 1 - \tan^2 y)\,dy$

$g(x,y) = \sin(x+y) + \cos(x-y),$

$dg = (\cos(x+y) - \sin(x-y))\;dx + (\cos(x+y) + \sin(x-y))\;dy$

e) $f(x,y) = e^{2x-3y}, \qquad df = e^{2x-3y}(2\,dx - 3\,dy)$

$g(x,y) = e^{x^2+y^2}, \qquad dg = 2\,e^{x^2+y^2}(x\,dx + y\,dy)$

f) $f(x,y) = \ln(x+y), \qquad df = \dfrac{dx+dy}{x+y}$

$g(x,y) = \ln(x^2+y^2), \qquad dg = \dfrac{2(x\,dx + y\,dy)}{x^2+y^2}$

$h(x,y) = \ln\left(1 + \dfrac{y}{x}\right), \qquad dh = -\dfrac{y\,dx}{x(x+y)} + \dfrac{dy}{x+y} = \dfrac{-y\,dx + x\,dy}{x(x+y)}$

g) $f(x,y) = \sqrt{x^2+y^2}, \qquad df = \dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$

$g(x,y) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}}, \qquad dg = -\dfrac{1}{2}\left(\dfrac{dx}{x\,\sqrt{x}} + \dfrac{dy}{y\,\sqrt{y}}\right)$