Funktionen mehrerer Variablen: Integralrechnung

Aufgaben mit Lösungen

Jörg Gayler, Lubov Vassilevskaya

Inhaltsverzeichnis

1. Doppelintegrale

1.1. Doppelintegrale mit konstanten Integrationsgrenzen

Berechnen Sie die folgenden Doppelintegrale

A1

Beispiel 1:

$$I = \int_{y=0}^{1} \int_{x=-2}^{1} (x^2 + y^2)\, dx\, dy$$

Innere Integration nach x: $$I = \int_{y=0}^{1} \int_{x=-2}^{1} (x^2 + y^2)\, dx\, dy = 3 \int_{0}^{1} (1 + y^2)\, dy \ = \ 4$$

Innere Integration nach y: $$I = \int_{y=0}^{1} \int_{x=-2}^{1} (x^2 + y^2)\, dx\, dy = \int_{-2}^{1} \left(x^2 + \frac{1}{3}\right) dx = 4$$

Aufgaben:

a) $I_1 = \int_{x=0}^{1} \int_{y=0}^{1} (x^2 + y)\, dx\, dy$, $\quad I_2 = \int_{x=0}^{1} \int_{y=0}^{3} \left(\sqrt{x} + \sqrt{y+1}\right) dx\, dy$, $\quad I_3 = \int_{x=0}^{1} \int_{y=0}^{1} \sqrt{xy}\, dx\, dy$

b) $I_1 = \int_{x=0}^{2} \int_{y=0}^{\pi} x \, \sin y \, dx\, dy$, $\quad I_2 = \int_{x=0}^{3} \int_{y=0}^{\pi} x^2 \, \sin y \, dx\, dy$, $\quad I_3 = \int_{x=0}^{1} \int_{y=0}^{\pi/4} x \, \cos(2y)\, dx\, dy$

c) $I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin(x + y)\, dx\, dy$, $\quad I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \cos(x + y)\, dx\, dy$, $\quad I_3 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} x \, \cos(x + y)\, dx\, dy$

A2

$$a)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\,\cos(2y)\,dx\,dy, \qquad I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin(2x)\,\cos(3y)\,dx\,dy$$

$$b)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\,\cos^2 y\,dx\,dy, \qquad I_2 = \int_{x=0}^{\pi/4}\int_{y=0}^{\pi/2} \sin(2x)\,\cos^2 y\,dx\,dy$$

$$c)\quad I_1 = \int_{x=0}^{\pi}\int_{y=1}^{2} y\cdot\cos(xy)\,dx\,dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{\pi} x\,\sin(xy)\,dx\,dy$$

$$d)\quad I_1 = \int_{x=1}^{3}\int_{y=1}^{2} x\,\ln(xy)\,dx\,dy, \qquad I_2 = \int_{x=1}^{3}\int_{y=1}^{2} x^2\,\ln(xy)\,dx\,dy$$

$$e)\quad I_1 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\sin y}{x}\,dx\,dy, \qquad I_2 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\cos y}{x^2}\,dx\,dy, \qquad I_3 = \int_{x=1}^{3}\int_{y=0}^{\pi/4} \frac{\cos(2y)}{x^3}\,dx\,dy$$

A3

$$a)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{1} e^{x-2y}\,dx\,dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{1} y^2\,e^{x+2}\,dx\,dy$$

$$b)\quad I_1 = \int_{x=0}^{1}\int_{y=1}^{2} \frac{x\,e^x}{y}\,dx\,dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=1}^{3} \frac{x\,e^x}{y^2}\,dx\,dy, \qquad I_3 = \int_{x=0}^{1}\int_{y=1}^{2} \frac{x\,e^{2x}}{y^3}\,dx\,dy$$

$$c)\quad I_1 = \int_{x=1}^{2}\int_{y=1}^{2} \left(\frac{2x}{y} - \frac{y}{x}\right)dx\,dy, \qquad I_2 = \int_{x=1}^{2}\int_{y=1}^{2} \left(\frac{x}{y} - \frac{y^2}{x^2}\right)dx\,dy$$

$$d)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{1} \frac{x}{1+xy}\,dx\,dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{1} \frac{x}{1+2xy}\,dx\,dy, \qquad I_3 = \int_{x=0}^{2}\int_{y=0}^{1} \frac{x^2}{1+xy}\,dx\,dy$$

$$e)\quad I_1 = \int_{x=0}^{4}\int_{y=0}^{1} \frac{\sqrt{x}}{1+y}\,dx\,dy, \qquad I_2 = \int_{x=0}^{4}\int_{y=0}^{1} \frac{\sqrt{x}}{(1+y)^2}\,dx\,dy$$

1.2. Doppelintegrale mit beliebigen Integrationsgrenzen

A4

$$a)\quad I_1 = \int_{y=0}^{1}\int_{x=0}^{y} xy\,dx\,dy, \qquad I_2 = \int_{y=0}^{2}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy, \qquad I_3 = \int_{x=0}^{1}\int_{y=0}^{\sqrt{4-x^2}} xy\,dy\,dx$$

$$b)\quad I_1 = \int_{x=0}^{3}\int_{y=0}^{x} x\,y^2\,dy\,dx, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{x} x^2\,y^2\,dy\,dx, \qquad I_3 = \int_{x=0}^{1}\int_{y=1-x}^{1-x^2} xy\,dy\,dx$$

$$c)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{x} (x^2+y^2)\,dy\,dx, \qquad I_2 = \int_{y=0}^{3}\int_{x=0}^{\sqrt{y}} (x^3+y^3)\,dx\,dy, \qquad I_3 = \int_{x=0}^{1}\int_{y=0}^{x^2-1} (x+y)\,dy\,dx$$

$$d)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{x} (1+\sin y)\,dy\,dx, \qquad I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{x} (\cos x+\sin y)\,dy\,dx$$

1.3. Doppelintegrale in Polarkoordinaten

Berechnen Sie die folgenden Doppelintegrale und zeichnen Sie den Integrationsbereich

A5

$$I_1 = \iint_A xy\,dx\,dy, \qquad A:\ 1 \leqslant r \leqslant 3,\quad 0 \leqslant \varphi \leqslant \frac{\pi}{4}$$

$$I_2 = \iint_A y^2\,\sqrt{4-x^2-y^2}\,dx\,dy, \qquad A:\ x^2+y^2 \leqslant 4,\quad y \geqslant 0$$

Berechnen Sie die folgenden Doppelintegrale

A6

$$a)\quad I_1 = \iint_A x^2\,e^{-(x^2+y^2)}\,dx\,dy, \qquad A = x^2+y^2 \leqslant 1$$

$$b)\quad I_1 = \int_{\varphi=0}^{\pi/2}\int_{r=0}^{\cos^2\varphi} r\,dr\,d\varphi,$$

1.4. Doppelintegrale in der Volumenberechnung

Berechnen Sie die Volumina der Körper, die durch folgende Flächen begrenzt werden oder durch andere Angaben bestimmt werden

A7

a) $f(x,y) = 2 + \sin x \cdot \sin y, \qquad A_f: \quad -\pi \leqslant x, y \leqslant \pi$

$g(x,y) = 2 + \sin x \cdot \sin y, \qquad A_g: \quad x^2 + y^2 \leqslant \pi^2$

b) $x^2 + y^2 = 9, \qquad z = 0, \qquad z = 9 - y$

c) $y = x^2, \quad y = 4, \quad z = 3 + x + 2y$

d) $z = 2 - 2x - y, \qquad x = 0, \qquad y = 0, \qquad z = 0$

2. Doppelintegrale: Lösungen

2.1. Doppelintegrale mit konstanten Integrationsgrenzen

L1

$$a)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{1} (x^2+y)\,dx\,dy = \int_0^1 \left(\frac{1}{3}+y\right) dy = \frac{5}{6}$$

$$I_2 = \int_{x=0}^{1}\int_{y=0}^{3} \left(\sqrt{x}+\sqrt{y+1}\right) dx\,dy = \int_0^3 \left(\frac{2}{3}+\sqrt{y+1}\right) dy = \frac{20}{3}$$

$$I_3 = \int_{x=0}^{1}\int_{y=0}^{1} \sqrt{xy}\,dx\,dy = \frac{2}{3}\int_0^1 \sqrt{y}\,dy = \frac{4}{9}$$

$$b)\quad I_1 = \int_{x=0}^{2}\int_{y=0}^{\pi} x\,\sin y\,dx\,dy = 2\int_0^{\pi} \sin y\,dy = 4$$

$$I_2 = \int_{x=0}^{3}\int_{y=0}^{\pi} x^2\,\sin y\,dx\,dy = 9\int_0^{\pi} \sin y\,dy = 18$$

$$I_3 = \int_{x=0}^{1}\int_{y=0}^{\pi/4} x\,\cos(2y)\,dx\,dy = \frac{1}{2}\int_0^{\pi/4} \cos(2y)\,dy = \frac{1}{4}$$

$$c)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin(x+y)\,dx\,dy = \int_0^{\pi/2} (\sin y+\cos y)\,dy = 2$$

$$I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \cos(x+y)\,dx\,dy = \int_0^{\pi/2} (-\sin y+\cos y)\,dy = 0$$

$$I_3 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} x\,\cos(x+y)\,dx\,dy = \int_0^{\pi/2} (-x\,\sin x + x\,\cos x)\,dx = -2+\frac{\pi}{2}$$

L2

a) $$I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\, \cos(2y)\, dx\, dy = \int_0^{\pi/2} \cos(2y)\, dy = 0$$

$$I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin(2x)\, \cos(3y)\, dx\, dy = \int_0^{\pi/2} \cos(3y)\, dy = -\frac{1}{3}$$

b) $$I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\, \cos^2 y\, dx\, dy = \int_0^{\pi/2} \cos^2 y\, dy = \frac{\pi}{4}$$

$$I_2 = \int_{x=0}^{\pi/4}\int_{y=0}^{\pi/2} \sin(2x)\, \cos^2 y\, dx\, dy = \frac{1}{2}\int_0^{\pi/2} \cos^2 y\, dy = \frac{\pi}{8}$$

c) $$I_1 = \int_{x=0}^{\pi}\int_{y=1}^{2} y \cdot \cos(xy)\, dx\, dy = \int_1^{2} \sin(\pi y)\, dy = -\frac{2}{\pi}$$

$$I_2 = \int_{x=0}^{2}\int_{y=0}^{\pi} x\, \sin(xy)\, dx\, dy = \int_0^{2} (1 - \cos(\pi x))\, dx = 2$$

d) $$I_1 = \int_{x=1}^{3}\int_{y=1}^{2} x\, \ln(xy)\, dx\, dy = \int_1^{3} (x \ln x - x + 2x \ln 2)\, dx = -6 + 8 \ln 2 + \frac{9}{2} \ln 3 \simeq 4.49$$

$$I_2 = \int_{x=1}^{3}\int_{y=1}^{2} x^2\, \ln(xy)\, dx\, dy = \int_1^{3} \left(x^2 \ln x - x^2 + 2x^2 \ln 2\right) dx =$$

$$= -\frac{104}{9} + \frac{52}{3} \ln 2 + 9 \ln 3 \simeq 10.37$$

e) $$I_1 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\sin y}{x}\, dx\, dy = \int_1^{2} \frac{dx}{x} = \ln 2 \simeq 0.69$$

$$I_2 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\cos y}{x^2}\, dx\, dy = \int_1^{2} \frac{dx}{x^2} = \frac{1}{2}$$

$$I_3 = \int_{x=1}^{3}\int_{y=0}^{\pi/4} \frac{\cos(2y)}{x^3}\, dx\, dy = \frac{1}{2}\int_1^{3} \frac{dx}{x^3} = \frac{2}{9}$$

L3

$$a)\quad I_1 = \int\limits_{x=0}^{1}\int\limits_{y=0}^{1} e^{x-2y}\,dx\,dy \;=\; \int\limits_0^1 \left(e^{1-2y} - e^{-2y}\right) dy \;=\; \frac{1}{2}\left(-1 + e + e^{-2} - e^{-1}\right) \simeq 0.74$$

$$I_2 = \int\limits_{x=0}^{2}\int\limits_{y=0}^{1} y^2\, e^{x+2}\,dy\,dx \;=\; \frac{1}{3}\int\limits_0^2 e^{x+2}\,dx = \frac{1}{3}\,(e^4 - e^2) \simeq 15.75$$

$$b)\quad I_1 = \int\limits_{x=0}^{1}\int\limits_{y=1}^{2} \frac{x\,e^x}{y}\,dy\,dx = \ln\,2\int\limits_0^1 x\,e^x\,dx \;=\ln 2 \simeq 0.69$$

$$I_2 = \int\limits_{x=0}^{2}\int\limits_{y=1}^{3} \frac{x\,e^x}{y^2}\,dy\,dx = \frac{2}{3}\int\limits_0^2 x\,e^x\,dx = \frac{2}{3}\left(1 + e^2\right)$$

$$I_3 = \int\limits_{x=0}^{1}\int\limits_{y=1}^{2} \frac{x\,e^{2x}}{y^3}\,dy\,dx = \frac{3}{8}\int\limits_0^1 x\,e^{2x}\,dx = \frac{3}{32}\left(1 + e^2\right)$$

$$c)\quad I_1 = \int\limits_{x=1}^{2}\int\limits_{y=1}^{2} \left(\frac{2x}{y} - \frac{y}{x}\right) dy\,dx = \frac{1}{2}\int\limits_1^2 \left(4\,x\ln 2\; - \frac{3}{x}\right) dx = \frac{3}{2}\,\ln 2 \simeq 1.04$$

$$I_2 = \int\limits_{x=1}^{2}\int\limits_{y=1}^{2} \left(\frac{x}{y} - \frac{y^2}{x^2}\right) dy\,dx = \frac{1}{3}\int\limits_1^2 \left(3\,\ln 2\cdot x - \frac{7}{x^2}\right) dx = -\frac{7}{6} + \frac{3}{2}\,\ln\,2 \simeq\; -0.13$$

$$d)\quad I_1 = \int\limits_{x=0}^{1}\int\limits_{y=0}^{1} \frac{x}{1+xy}\,dy\,dx = \int\limits_0^1 \ln(1+x)\,dx = 2\,\ln\,2 - 1 \simeq 0.39$$

$$I_2 = \int\limits_{x=0}^{2}\int\limits_{y=0}^{1} \frac{x}{1+2xy}\,dy\,dx = \frac{1}{2}\int\limits_0^2 \ln(1+2x)\,dx = \frac{5}{4}\,\ln\,5 - 1 \simeq 1.01$$

$$I_3 = \int\limits_{x=0}^{2}\int\limits_{y=0}^{1} \frac{x^2}{1+xy}\,dy\,dx = \int\limits_0^2 x\,\ln(1+x)\,dx = \int\limits_1^3 (u-1)\,\ln u\,du \;=\; \frac{3}{2}\,\ln\,3 \simeq 1.65\;\;(u = 1+x)$$

$$e)\quad I_1 = \int\limits_{x=0}^{4}\int\limits_{y=0}^{1} \frac{\sqrt{x}}{1+y}\,dy\,dx = \ln\,2\int\limits_0^4 \sqrt{x}\,dx = \frac{16}{3}\,\ln\,2 \simeq 3.70$$

$$I_2 = \int\limits_{x=0}^{4}\int\limits_{y=0}^{1} \frac{\sqrt{x}}{(1+y)^2}\,dy\,dx = \frac{1}{2}\int\limits_0^4 \sqrt{x}\,dx = \frac{8}{3}$$

2.2. Doppelintegrale mit beliebigen Integrationsgrenzen

L4

a) $$I_1 = \int_{y=0}^{1} \int_{x=0}^{y} xy\,dx\,dy = \frac{1}{2} \int_0^1 y^3\,dy = \frac{1}{8}$$

$$I_2 = \int_{y=0}^{2} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy = \frac{1}{2} \int_0^2 y^2\,dy = \frac{4}{3}$$

$$I_3 = \int_{x=0}^{1} \int_{y=0}^{\sqrt{4-x^2}} xy\,dx\,dy = \frac{1}{2} \int_0^1 x\,(4-x^2)\,dx = \frac{7}{8}$$

b) $$I_1 = \int_{x=0}^{3} \int_{y=0}^{x} x\,y^2\,dx\,dy = \frac{1}{3} \int_0^3 x^4\,dx = \frac{81}{5}$$

$$I_2 = \int_{x=0}^{2} \int_{y=0}^{x} x^2\,y^2\,dx\,dy = \frac{1}{3} \int_0^2 x^5\,dx = \frac{32}{9}$$

$$I_3 = \int_{x=0}^{1} \int_{y=1-x}^{1-x^2} xy\,dx\,dy = \frac{1}{2} \int_0^1 x\left((1-x^2)^2 - (1-x)^2\right) dx = \frac{1}{24}$$

c) $$I_1 = \int_{x=0}^{1} \int_{y=0}^{x} (x^2+y^2)\,dx\,dy = \frac{4}{3} \int_0^1 x^3\,dx = \frac{1}{3}$$

$$I_2 = \int_{y=0}^{3} \int_{x=0}^{\sqrt{y}} (x^3+y^3)\,dx\,dy = \int_0^3 \left(\frac{y^2}{4} + y^{7/2}\right) dy = \frac{9}{4} + 18\sqrt{3} \simeq 33.43$$

$$I_3 = \int_{x=0}^{1} \int_{y=0}^{x^2-1} (x+y)\,dx\,dy = \int_0^1 \left(x\,(x^2-1) + \frac{1}{2}\,(x^2-1)^2\right) dx = \frac{1}{60}$$

d) $$I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^{x} (1+\sin y)\,dx\,dy = \int_0^{\pi/2} (1+x-\cos x)\,dx = \frac{\pi^2}{8} + \frac{\pi}{2} - 1 = 1.8$$

$$I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^{x} (\cos x + \sin y)\,dx\,dy = \int_0^{\pi/2} (1 + x\,\cos x - \cos x)\,dx = -2 + \pi \simeq 1.14$$

2.3. Doppelintegrale in Polarkoordinaten

L5

$$I_1 = \iint_A xy\,dx\,dy = \frac{1}{2}\int_{r=1}^{3}\int_{\varphi=0}^{\pi/4} r^3\,\sin(2\varphi)dr\,d\varphi = 10\int_{\varphi=0}^{\pi/4}\sin(2\varphi)d\varphi = 5$$

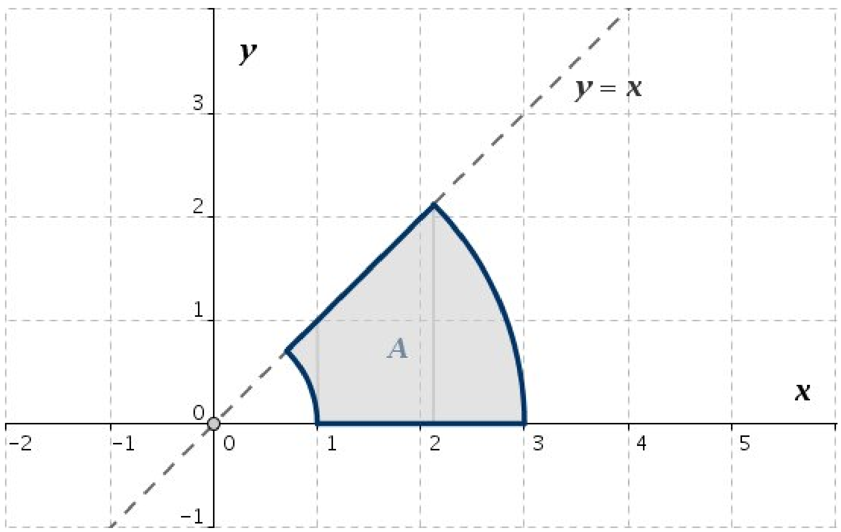

Abbildung 1: Darstellung des Integrationsbereiches für das Integral I_1 $A: \; 1 \leqslant r \leqslant 3, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{4}$

$$I_2 = \int_A y^2\,\sqrt{4 - x^2 - y^2}\,dx\,dy = \int_{r=0}^{2} r^3\,\sqrt{4 - r^2}\,dr\int_{\varphi=0}^{\pi}\sin^2\,\varphi\,d\varphi = \frac{\pi}{2}\int_{r=0}^{2} r^3\,\sqrt{4 - r^2}\,dr = \frac{32}{15}\pi \simeq 6.702,$$

$u = 4 - r^2$

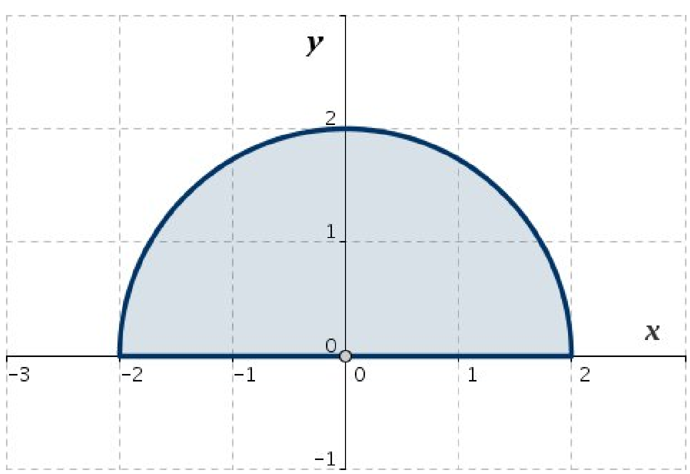

Abbildung 2: Darstellung des Integrationsbereiches für das Integral I_2 $A: \; 0 \leqslant r \leqslant 2, \quad 0 \leqslant \varphi \leqslant \pi$

L6

a) $$I_1 = \iint_A x^2\, e^{-(x^2+y^2)}\, dx\, dy = \int_{\varphi=0}^{2\pi} \cos^2\varphi\, d\varphi \int_{r=0}^{1} r^3\, e^{-r^2}\, dr = \pi \int_{r=0}^{1} r^3\, e^{-r^2}\, dr = \pi\left(\frac{1}{2} - e^{-1}\right) \simeq 0.415$$

b) $$I_1 = \int_{\varphi=0}^{\pi/2} \int_{r=0}^{\cos^2\varphi} r\, dr\, d\varphi = \frac{1}{2}\int_{\varphi=0}^{\pi/2} \cos^4\varphi d\varphi = \frac{1}{16}\int_{\varphi=0}^{\pi/2} (\cos(4\varphi) + 4\cos(2\varphi) + 3)\, d\varphi = \frac{3}{32}\pi \simeq 0.295$$

2.4. Doppelintegrale in der Volumenberechnung

L7

a) $f(x,y) = 2 + \sin x \cdot \sin y, \qquad A_f: \ -\pi \leqslant x, y \leqslant \pi$

$$V = \iint_A (2 + \sin x \cdot \sin y)\, dx\, dy = \int_{x=-\pi}^{\pi} \int_{y=-\pi}^{\pi} (2 + \sin x \cdot \sin y)\, dx\, dy = 4\pi \int_{y=-\pi}^{\pi} dy = 8\pi^2 \simeq 78.96 \text{ VE}$$

$g(x,y) = 2 + \sin x \cdot \sin y, \qquad A_g: \ x^2 + y^2 \leqslant \pi^2$

$$V = \iint_A (2 + \sin x \cdot \sin y)\, dx\, dy = \int_{x=-\pi}^{\pi} \int_{y=-\sqrt{\pi^2-x^2}}^{\sqrt{\pi^2-x^2}} (2 + \sin x \cdot \sin y)\, dy\, dx =$$

$$= 4\int_{x=-\pi}^{\pi} \sqrt{\pi^2 - x^2}\, dx = 2\pi^3 \simeq 62.01 \text{ VE}$$

b) $x^2 + y^2 = 9, \qquad z = 0, \qquad z = 9 - y$

$$V = \iint_A (9 - y)\, dx\, dy = 81\pi \simeq 254.47 \text{ VE}$$

c) $y = x^2, \quad y = 4, \quad z = 3 + x + 2y$

$$V = \iint_A (3 + x + 2y)\, dxdy = \int_{x=-2}^{2} \int_{y=x^2}^{4} (3 + x + 2y)\, dy\, dx = \frac{416}{5} \simeq 83.2 \text{ VE}$$

d) $z = 2 - 2x - y, \qquad x = 0, \qquad y = 0, \qquad z = 0, \qquad V = \dfrac{2}{3}$