Von Studenten vorgeschlagene Integrationsbereiche

1-E2

Bestimmen Sie die Integrationsgrenzen für das folgende Doppelintegral

$$\iint_A f(x, y)\, dA$$

Aufgabe 1: $\quad A: \quad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}, \qquad 0 \leqslant y \leqslant 2 + \sin x$

Aufgabe 2: $\quad g_1(x) = \frac{x^2}{4} - 3, \qquad g_2(x) = \frac{1}{2}\cos^2(3x) + \frac{5}{2}$

Aufgabe 3:

$$f_1(x) = \frac{3}{2}x + 3, \qquad f_2(x) = -\frac{2}{3}x + 3, \qquad g(x) = \frac{x^2}{2} - 2$$

Aufgabe 4:

$$g_1(x) = \sqrt{9 - x^2}, \qquad g_2(x) = \sqrt{x}, \qquad y = 0, \qquad x > 0$$

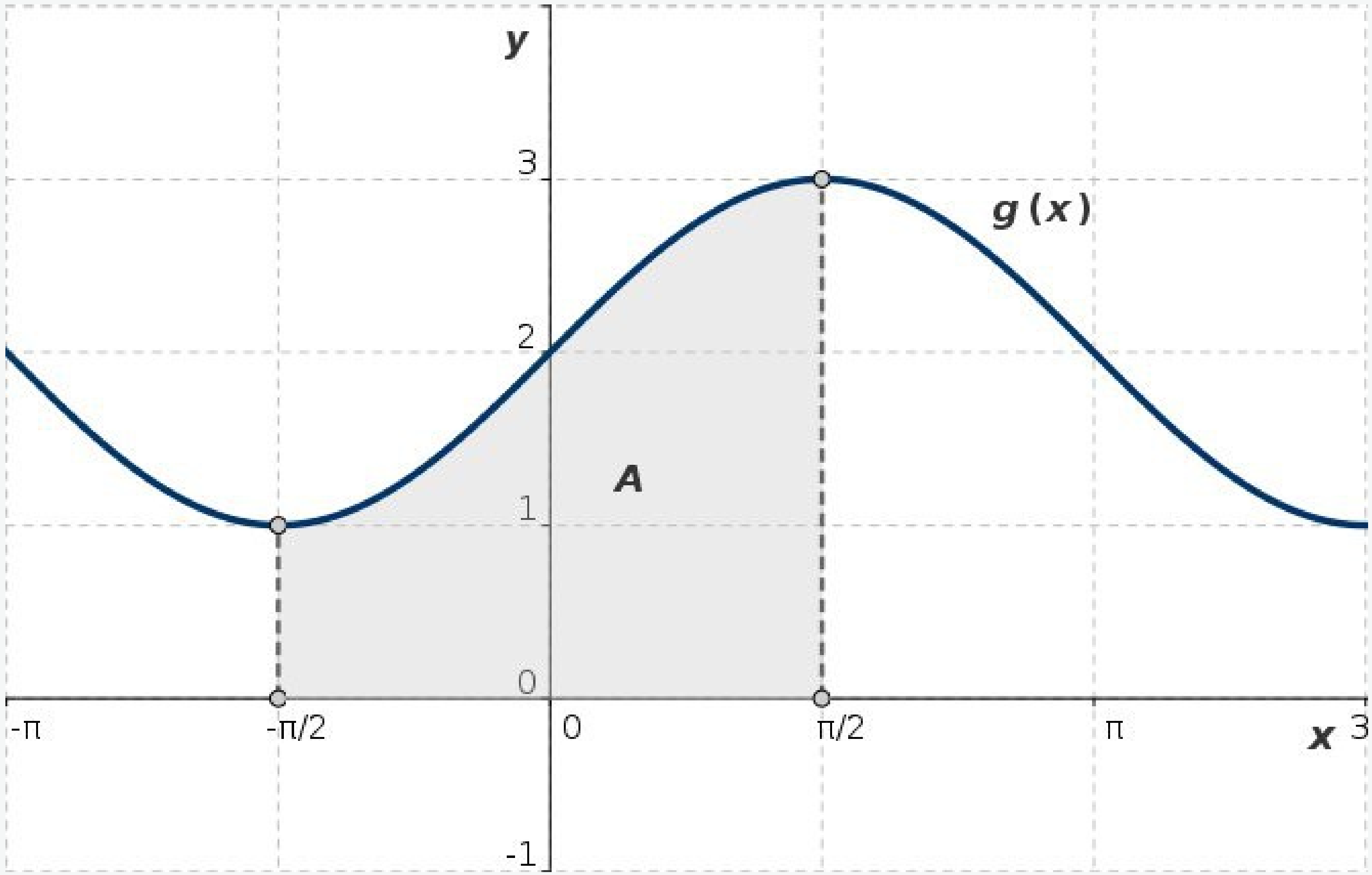

Abb. 1-1: Zur Bestimmung der Integrationsgrenzen (innere Integration nach y), vorgeschlagen von Michel Jürgensen

$$A: \quad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}, \qquad 0 \leqslant y \leqslant g(x), \qquad g(x) = 2 + \sin x$$

$$I = \iint_A f(x, y)\, dx\, dy = \int_{x=-\pi/2}^{\pi/2} \int_{y=0}^{2+\sin x} f(x, y)\, dy\, dx$$

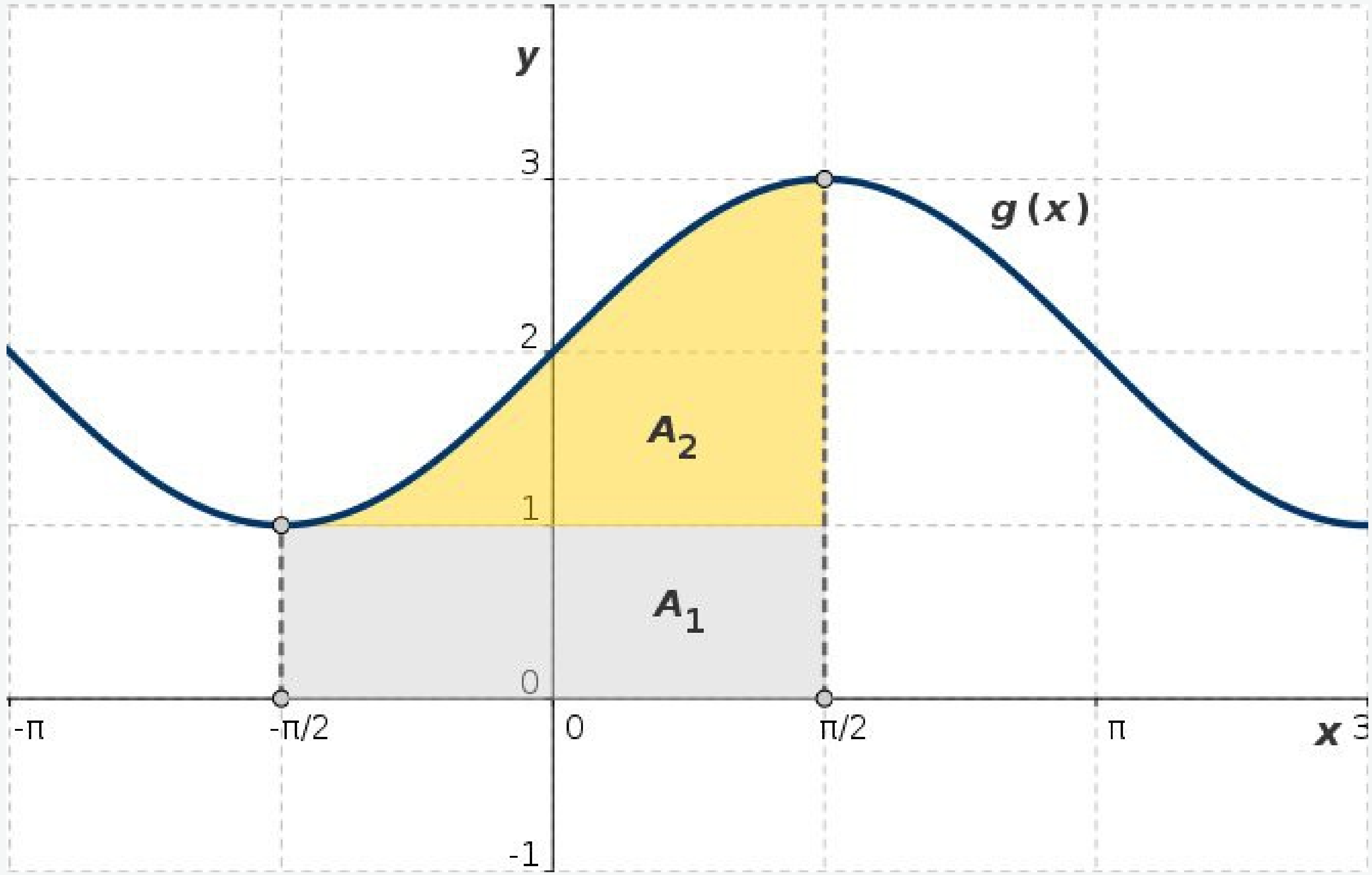

Abb. 1-2: Zur Bestimmung der Integrationsgrenzen (innere Integration nach x)

$$A = A_1 + A_2$$

$$I = \int\limits_{y=0}^{1} \int\limits_{x=-\pi/2}^{\pi/2} f(x, y)\, dx\, dy + \int\limits_{y=1}^{3} \int\limits_{x=\arcsin(y-2)}^{\pi/2} f(x, y)\, dx\, dy$$

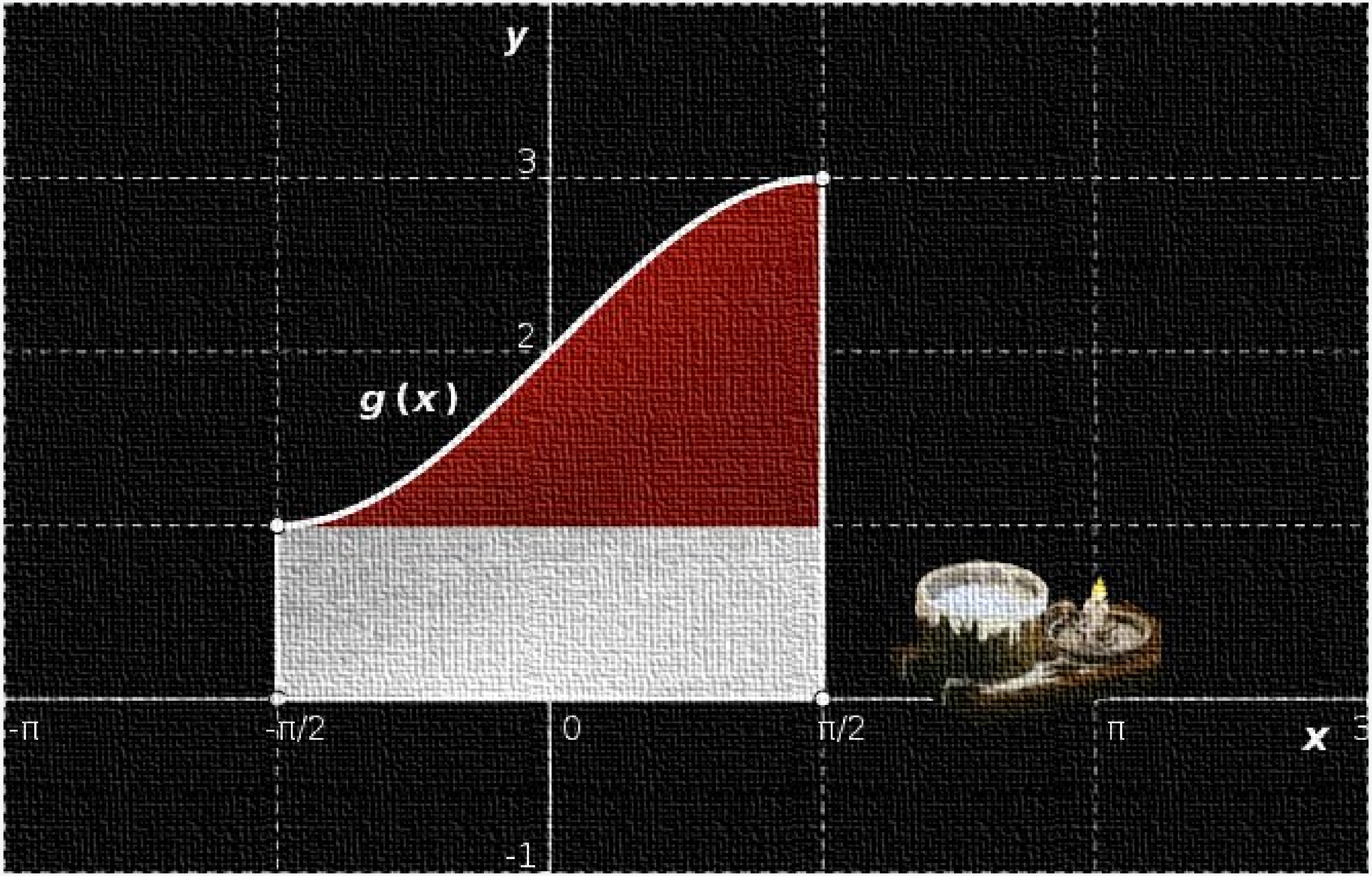

Abb. 1-3: Der Integrationsbereich der Aufgabe

$$f(x) = 2 + \sin x$$

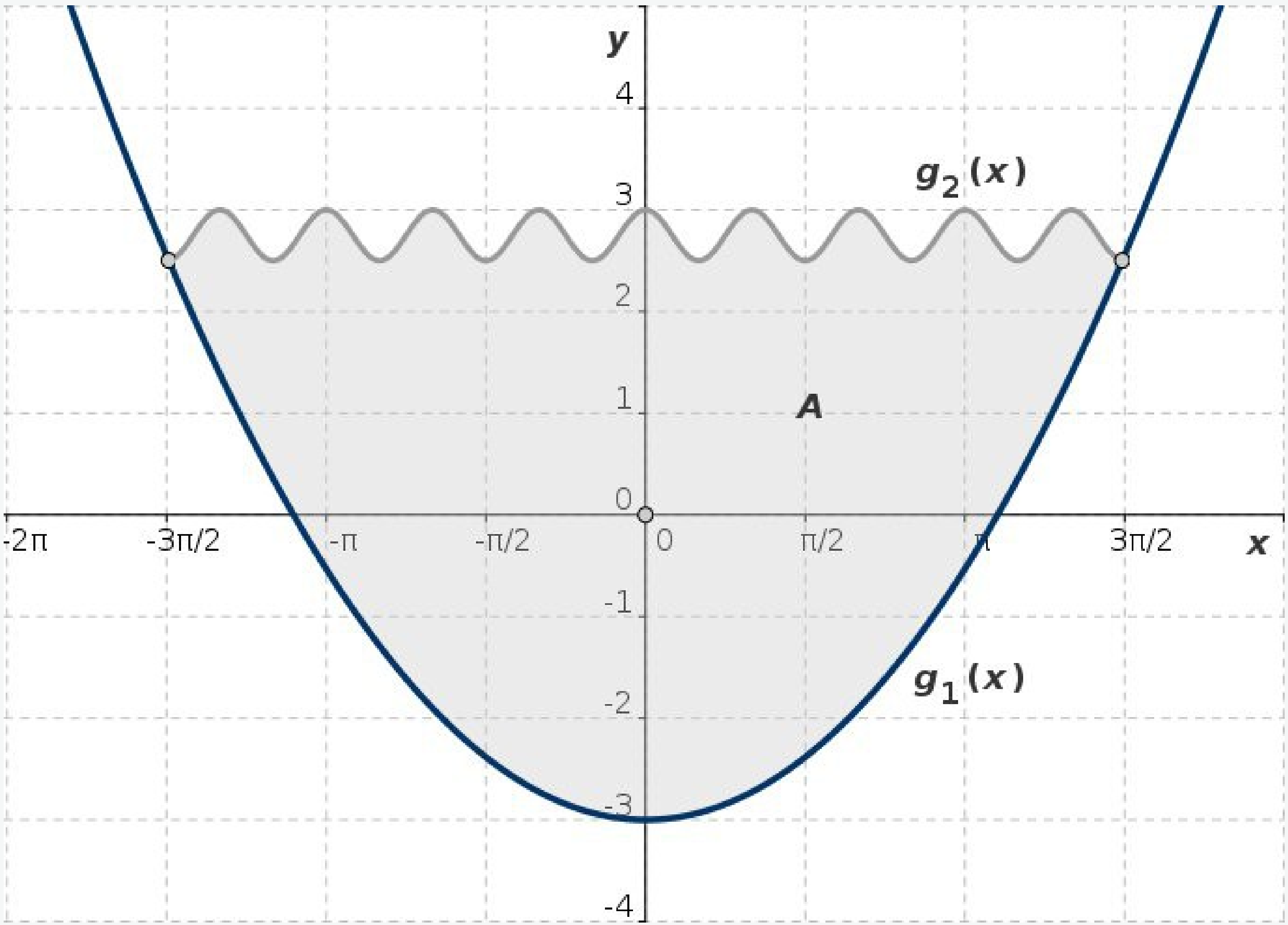

Abb. 2-1: Der Integrationsbereich der Aufgabe, vorgeschlagen von Konstantin Lühe

$$g_1(x) = \frac{x^2}{4} - 3, \qquad g_2(x) = \frac{1}{2}\cos^2(3\,x) + \frac{5}{2}$$

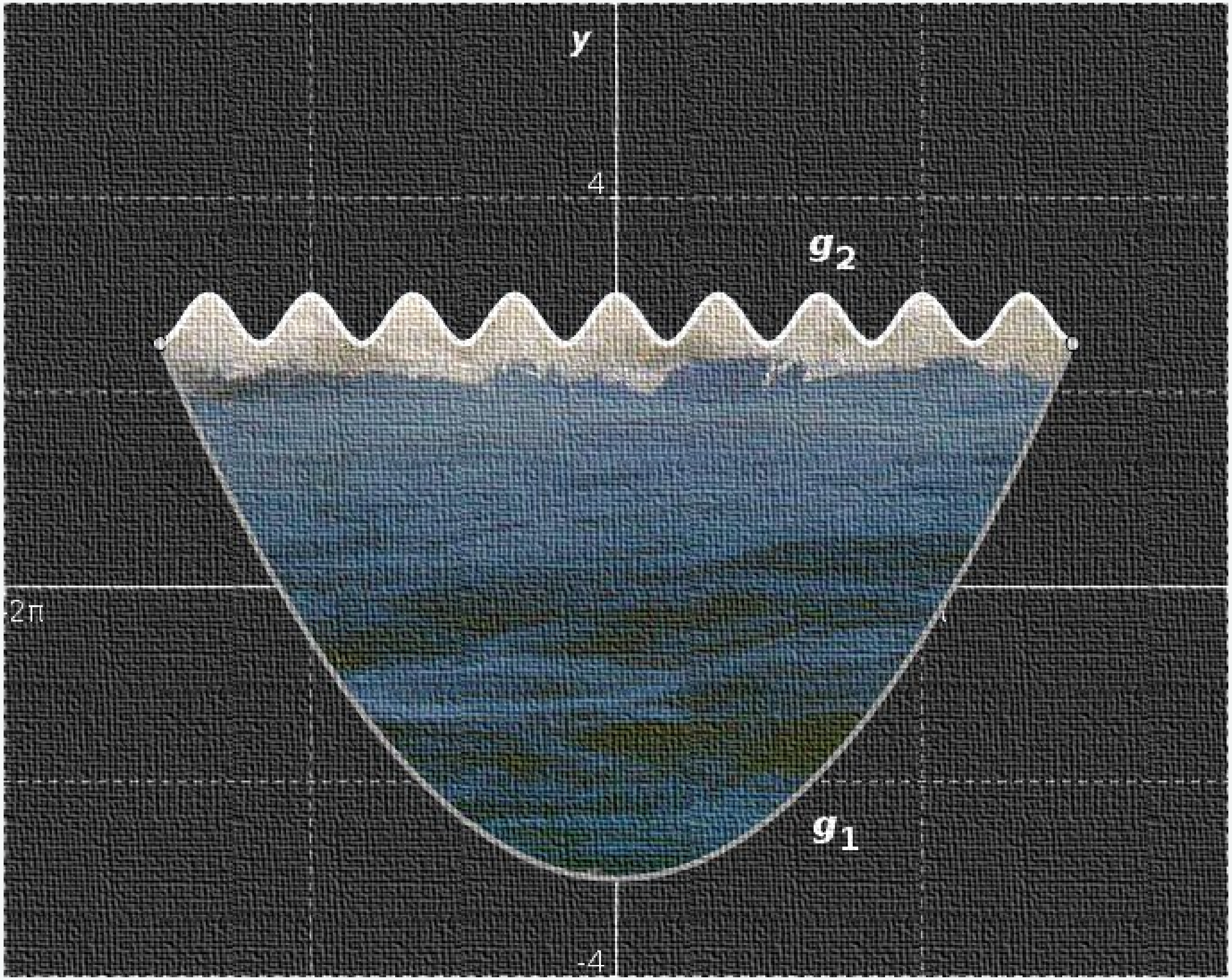

Abb. 2-1: Der Integrationsbereich der Aufgabe

$$I = \int_{x=-4}^{4} \int_{y=x^2/4-3}^{\frac{1}{2}\cos^2(3x) + \frac{5}{2}} f(x, y)\, dy\, dx$$

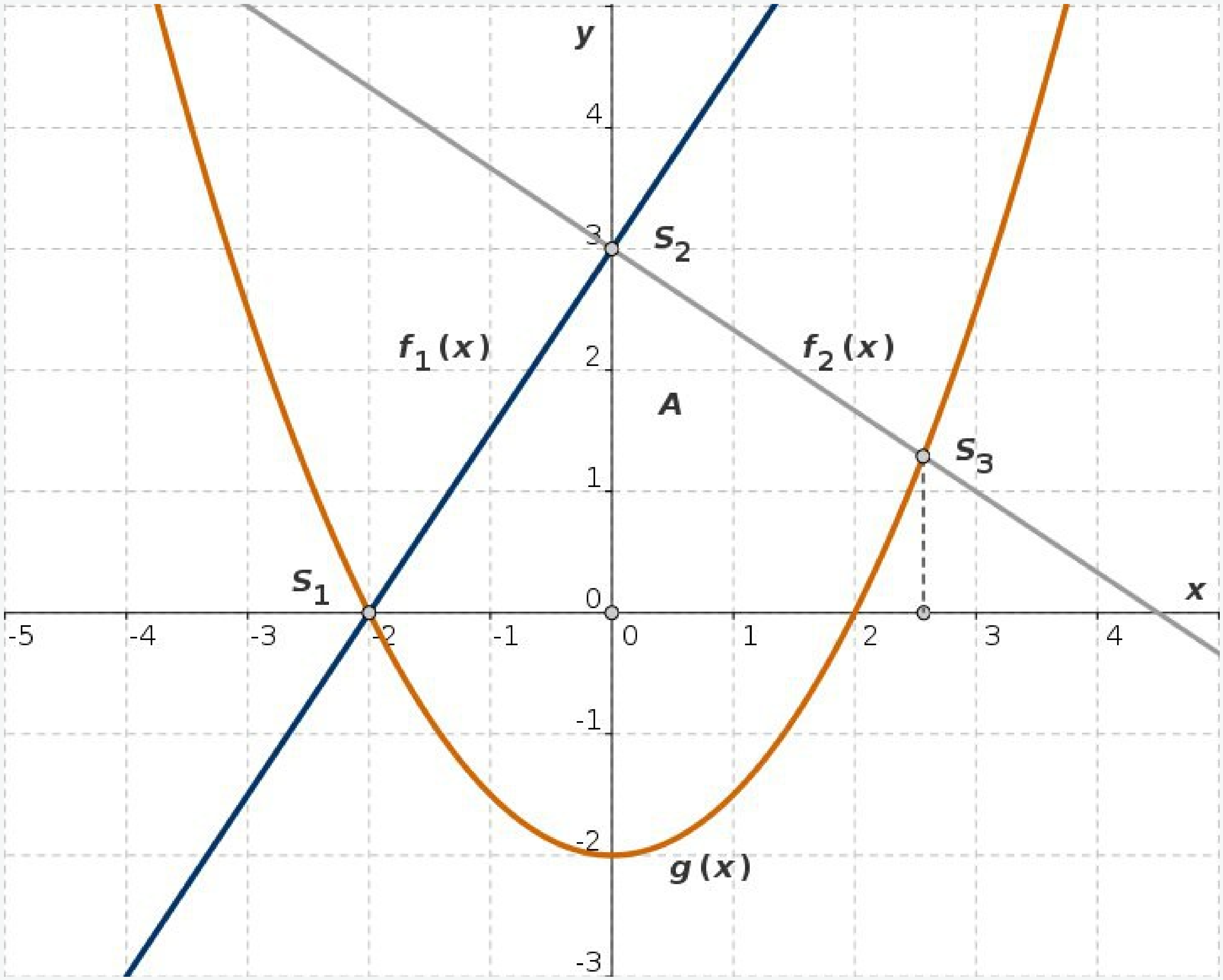

Abb. 3-1: Der Integrationsbereich der Aufgabe, vorgeschlagen von Thomas Nieber

Integrationsgrenzen: Lösung 3

$$f_1(x) = \frac{3}{2}\,x + 3, \qquad f_2(x) = -\frac{2}{3}\,x + 3, \qquad g(x) = \frac{x^2}{2} - 2$$

$$f_1(x) = g(x) \qquad S_1 = (-2,\ 0)$$

$$f_1(x) = f_2(x) \qquad S_2 = (0,\ 3)$$

$$f_2(x) = g(x) \qquad S_3 = (2.57,\ 1.29)$$

Der Integrationsbereich der Aufgabe (innere Integration nach y)

$$A = A_1 + A_2$$

$$A_1 : \quad -2 \leqslant x \leqslant 0, \qquad g(x) \leqslant y \leqslant f_1(x)$$

$$A_2 : \quad 0 \leqslant x \leqslant 2.57, \qquad g(x) \leqslant y \leqslant f_2(x)$$

$$I = \int\limits_{x = x_{S_1}}^{0} \int\limits_{y = g(x)}^{f_1(x)} f(x, y)\, dx\, dy + \int\limits_{x = 0}^{x_{S_3}} \int\limits_{y = g(x)}^{f_2(x)} f(x, y)\, dx\, dy =$$

$$= \int\limits_{x = -2}^{0} \int\limits_{y = \frac{x^2}{2} - 2}^{\frac{3}{2}x + 3} f(x, y)\, dy\, dx + \int\limits_{x = 0}^{2.57} \int\limits_{y = \frac{x^2}{2} - 2}^{-\frac{2}{3}x + 3} f(x, y)\, dy\, dx$$

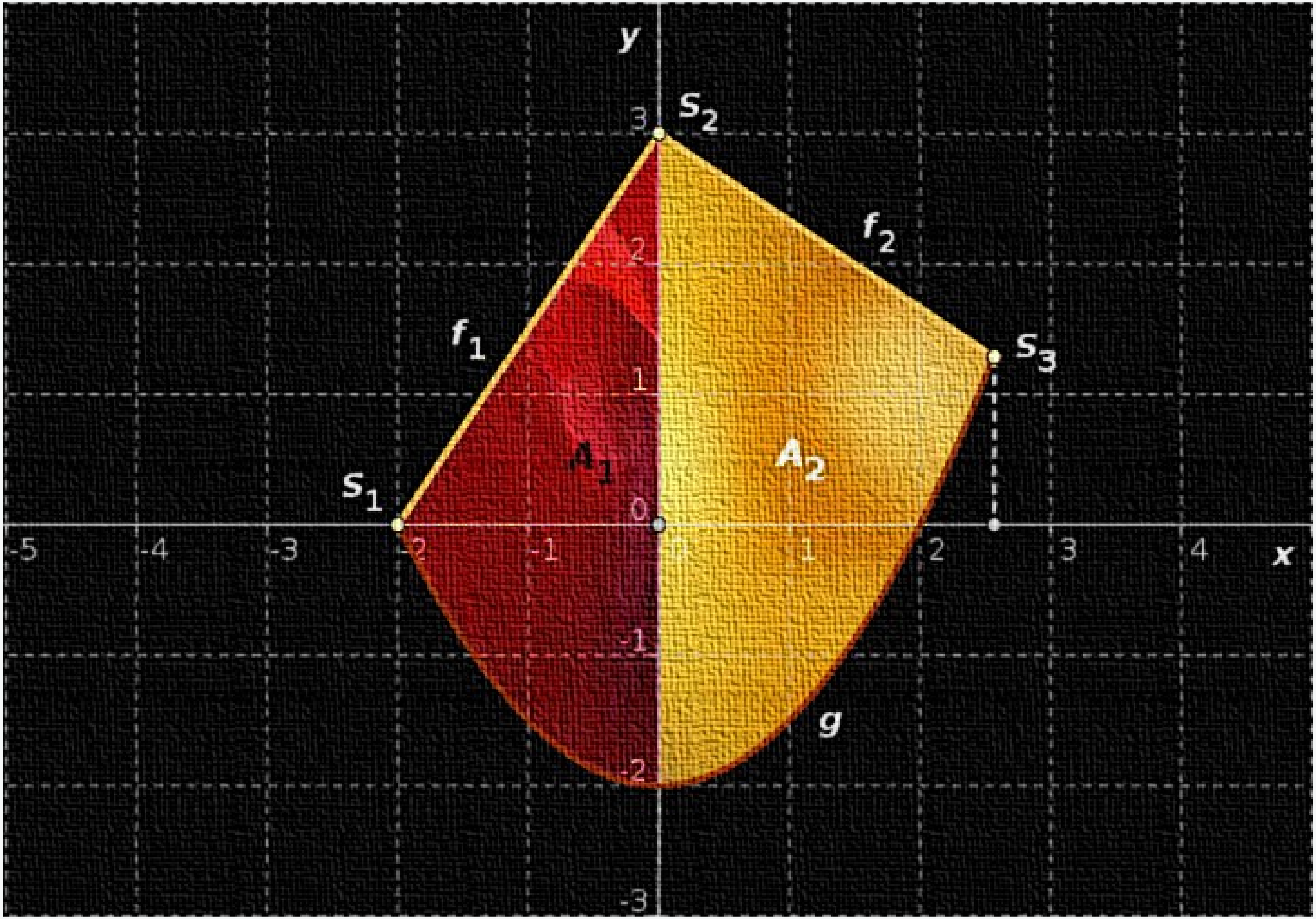

Abb. 3-2: Der Integrationsbereich der Aufgabe (innere Integration nach y)

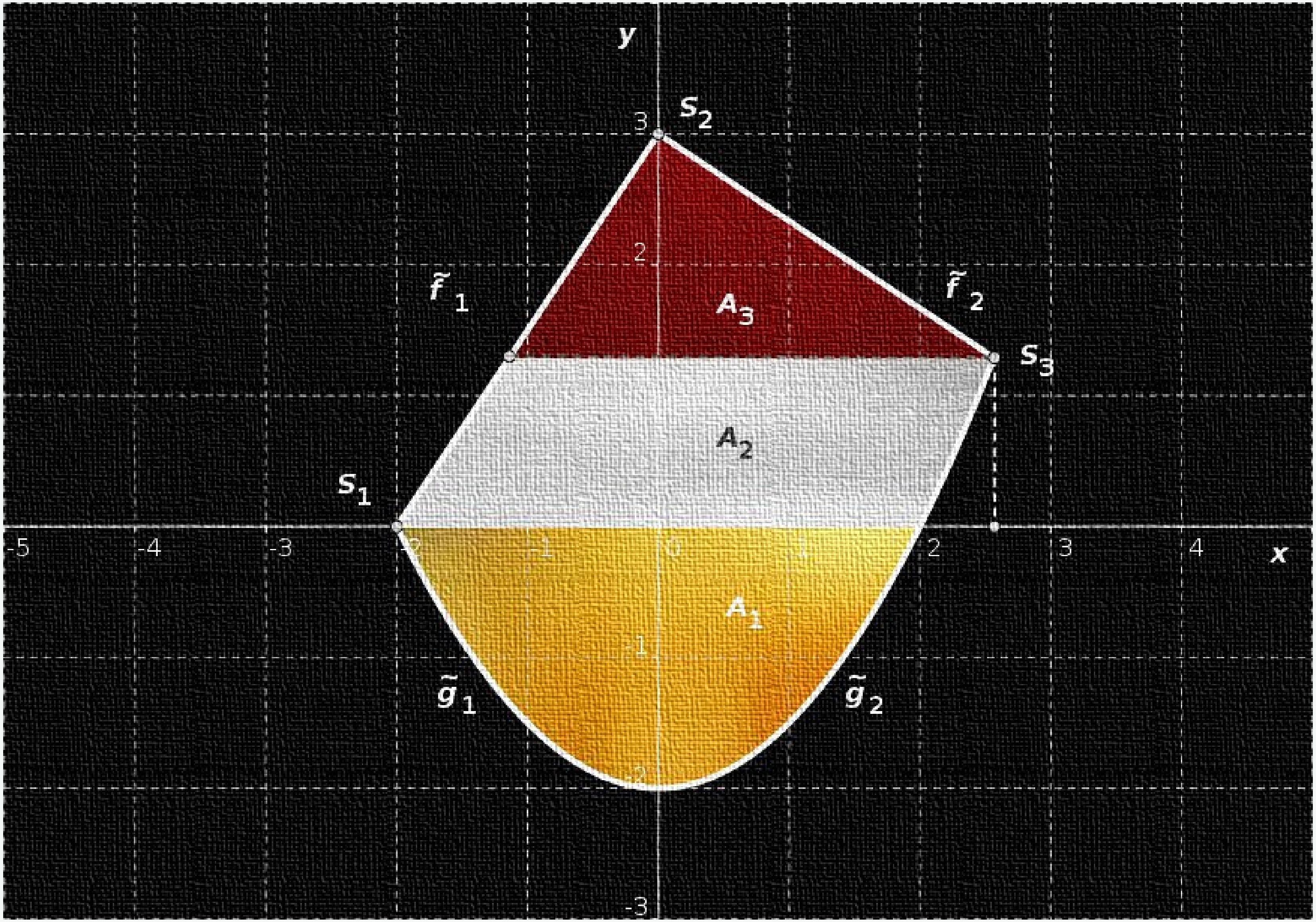

Abb. 3-3: Der Integrationsbereich der Aufgabe (innere Integration nach x)

$$f_1(x): \quad y = \frac{3}{2}\, x + 3 \quad \Rightarrow \quad x = \frac{2}{3}\, y - 2, \qquad \tilde{f}_1(y) = \frac{2}{3}\, y - 2$$

$$f_2(x): \quad y = -\frac{2}{3}\, x + 3 \quad \Rightarrow \quad x = \frac{9}{2} - \frac{3}{2}\, y, \qquad \tilde{f}_2(y) = \frac{9}{2} - \frac{3}{2}\, y$$

$$g(x): \quad y = \frac{x^2}{2} - 2, \quad \Rightarrow \quad x = \pm\sqrt{2y + 4}$$

$$\tilde{g}_1(y) = -\sqrt{2y + 4}, \qquad \tilde{g}_2(y) = \sqrt{2y + 4}$$

$$I = \int\limits_{y=-2}^{0} \int\limits_{x=\tilde{g}_1(y)}^{\tilde{g}_2(y)} f(x, y)\, dy\, dx + \int\limits_{y=0}^{y_{S_3}} \int\limits_{x=\tilde{f}_1(y)}^{\tilde{g}_2(y)} f(x, y)\, dy\, dx +$$

$$+ \int\limits_{y=y_{S_3}}^{y_{S_2}} \int\limits_{x=\tilde{f}_1(y)}^{\tilde{f}_2(y)} f(x, y)\, dy\, dx$$

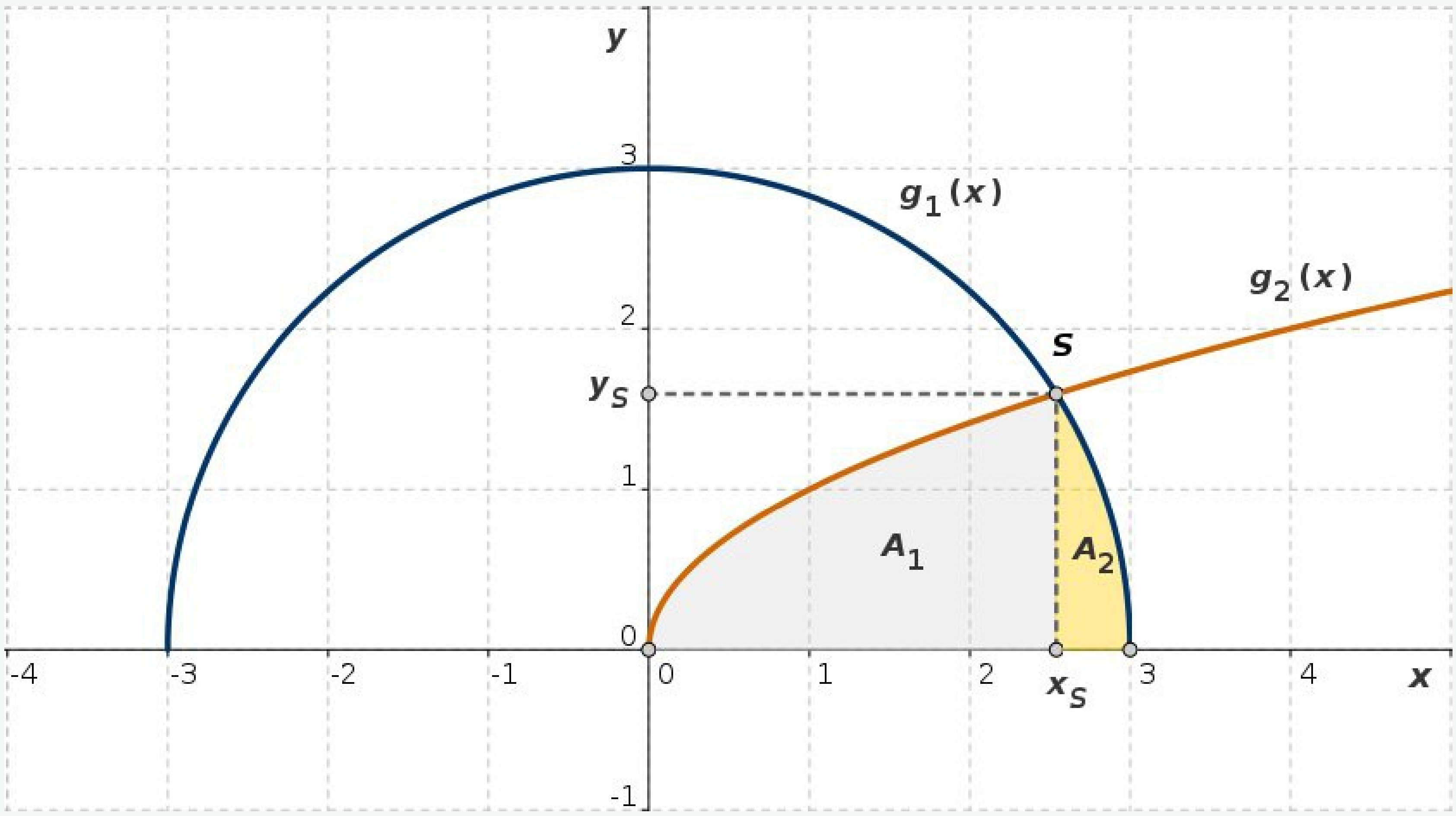

Abb. 4-2: Der Integrationsbereich der Aufgabe (innere Integration nach y), vorgeschlagen von Sebastian Stang

$$g_1(x) = \sqrt{9 - x^2}\,, \qquad g_2(x) = \sqrt{x}$$

$$I = \int_{x=0}^{x_S} \int_{y=0}^{g_2(x)} f(x, y)\, dy\, dx + \int_{x=x_S}^{3} \int_{y=0}^{g_1(x)} f(x, y)\, dy\, dx$$

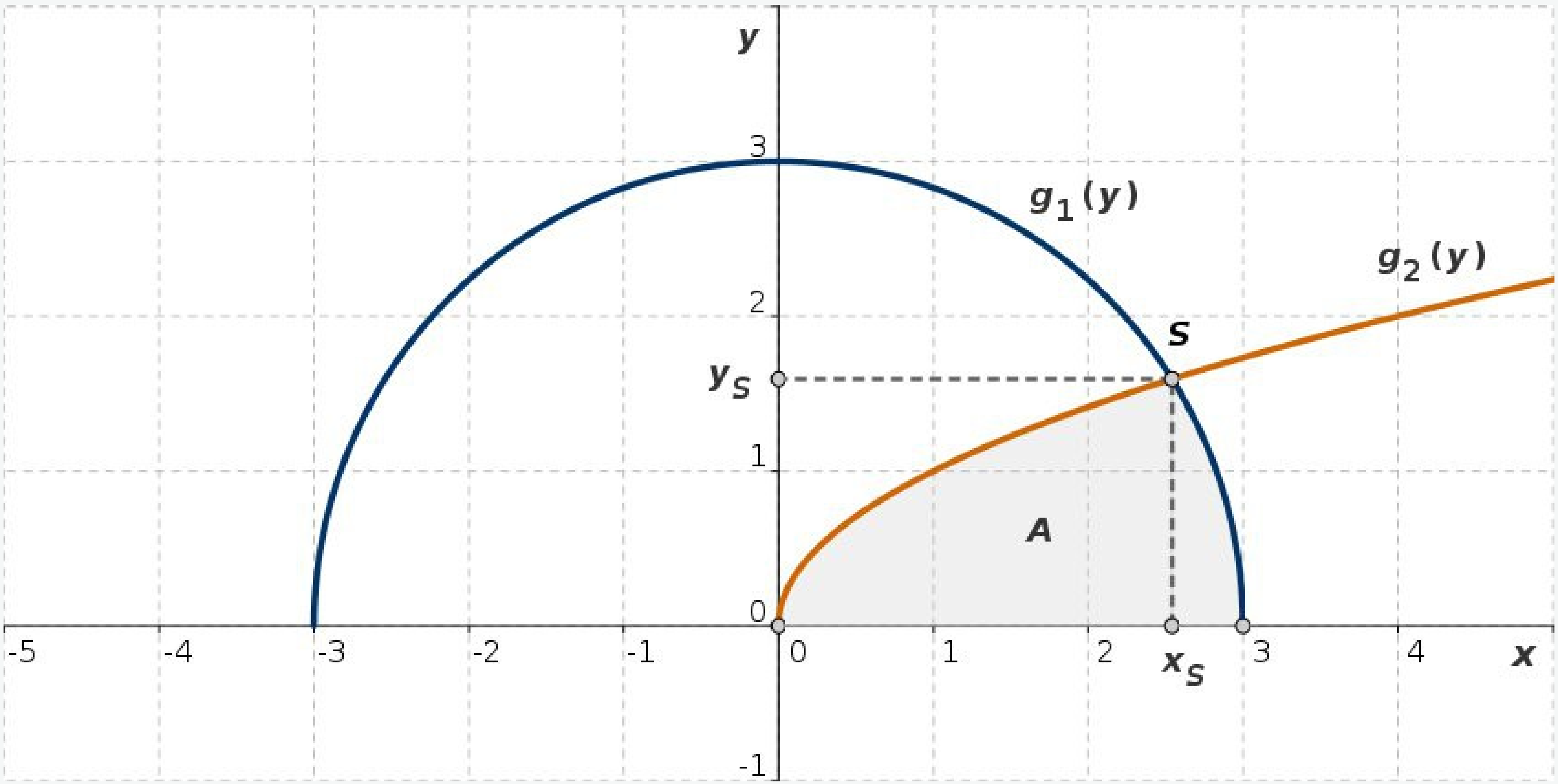

Abb. 4-1: Der Integrationsbereich der Aufgabe (innere Integration nach x)

$$g_1(y) = \sqrt{9 - y^2}\,, \qquad g_2(y) = y^2$$

$$I = \int_{y=y_S}^{0} \int_{x=g_2(y)}^{g_1(y)} f(x, y)\, dx\, dy$$

Zur Bestimmung des Schnittpunktes S

$$g_1(x) = \sqrt{9 - x^2}\,, \qquad g_2(x) = \sqrt{x}$$

$$g_1(x) = g_2(x) \quad \Leftrightarrow \quad \sqrt{9 - x^2} = \sqrt{x}$$

$$\left(\sqrt{9 - x^2}\right)^2 = \left(\sqrt{x}\right)^2 \quad \Leftrightarrow \quad 9 - x^2 = x \quad \Leftrightarrow \quad x^2 + x - 9 = 0$$

$$x_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 9} \quad \Rightarrow$$

$$x_S = -\frac{1}{2} + \sqrt{\frac{1}{4} + 9} = \frac{-1 + \sqrt{37}}{2} \qquad (x_S > 0)$$

$$S = (x_S\,,\; y_S = g(x_S))$$