Ein nicht-sinusförmiger Schwingungsvorgang $u = u\,(t)$ mit der Kreisfrequenz $\omega_0$ und der Schwingungsdauer (Periodendauer) $T$ kann wie folgt nach Fourier in seine harmonischen Bestandteile (Grundschwingung und Oberschwingungen) zerlegt werden:

$$f\,(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cdot \cos\,(n\,\omega_0\,t) + b_n \cdot \sin\,(n\,\omega_0\,t) \right]$$

Dabei bedeuten:

$\omega_0 = 2\pi / T$ – Kreisfrequenz der Grundschwingung

$n\,\omega_0$ – Kreisfrequenzen der harmonischen Oberschwingungen

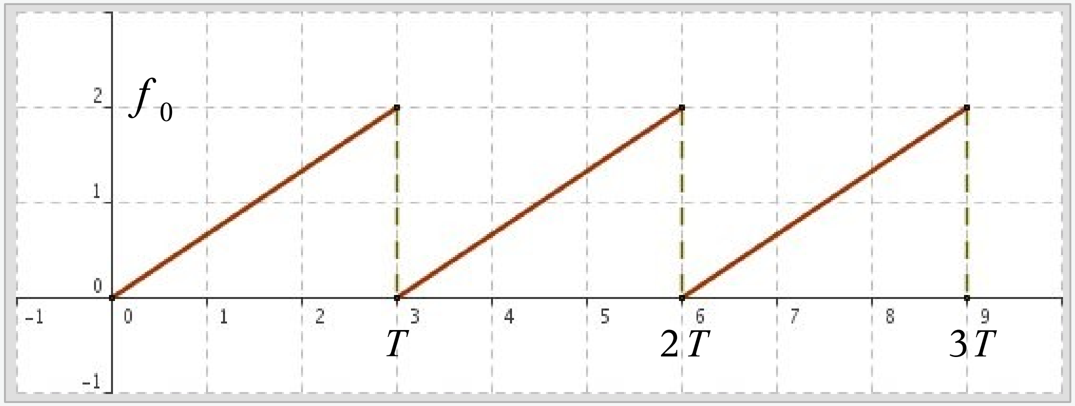
Die Fourierkoeffizienten dieser Zerlegung werden dabei aus den Integralformeln berechnet:

$$a_0 = \frac{2}{T} \cdot \int_0^T u\,(t)\; dt$$

$$a_n = \frac{2}{T} \cdot \int_0^T u\,(t) \cdot \cos\,(n\,\omega_0\,t)\; dt$$

$$b_n = \frac{2}{T} \cdot \int_0^T u\,(t) \cdot \sin\,(n\,\omega_0\,t)\; dt$$

Kippschwingung (Sägezahnimpuls)




$$f(t) = \frac{f_0}{T}\, t \qquad (0 \leqslant t < T)$$

Berechnen Sie die Fourierkoeffizienten:

$$a_0 = \frac{2}{T} \cdot \int_0^T f(t)\, dt$$

$$a_n = \frac{2}{T} \cdot \int_0^T f(t) \cdot \cos(n\,\omega_0 t)\, dt = \frac{2\,u_0}{T^2} \cdot \int_0^T t\, \cos(n\,\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \cdot \int_0^T f(t) \cdot \sin(n\,\omega_0 t)\, dt = \frac{2\,u_0}{T^2} \cdot \int_0^T t\, \sin(n\,\omega_0 t)\, dt$$

Berechnung des Fourierkoeffizienten $a_0$:

$$a_0 = \frac{2}{T} \cdot \int_0^T f(t)\, dt = \frac{2}{T} \cdot \int_0^T \frac{f_0}{T}\, t\, dt = \frac{2 f_0}{T^2} \cdot \int_0^T t\, dt = f_0$$

Berechnung der Fourierkoeffizienten $a_n$:

$$a_n = \frac{2 f_0}{T^2} \cdot \int_0^T t \cdot \cos(n \omega_0 t)\, dt = \frac{2 f_0}{T^2\, n^2\, \omega_0^2} \cdot \Big[ \cos(n \omega_0 t) + n \omega_0 t \cdot \sin(n \omega_0 t) \Big]_0^T =$$

$$\boxed{\int x \cos(a x)\, dx = \frac{1}{a^2} \left[ \cos(a x) + a x \sin(a x) \right]}$$

$$= \frac{2 f_0}{T^2 n^2 \omega_0^2} \Big[ \cos(n \omega_0 T) + n \omega_0 T \cdot \sin(n \omega_0 T) - 1 \Big] \overset{\omega_0 T = 2\pi}{=}$$

$$= \frac{2 f_0}{T^2 n^2 \omega_0^2} \Big[ 1 - 1 \Big] = 0$$

Berechnung der Fourierkoeffizienten $b_n$:

$$b_n = \frac{2 f_0}{T^2} \cdot \int_0^T t \sin(n \omega_0 t)\, dt = \frac{2 f_0}{T^2} \cdot \frac{1}{n^2 \omega_0^2} \left[ \sin(n \omega_0 t) - t\, n\, \omega_0 \cdot \cos(n \omega_0 t) \right]_0^T =$$

$$= \frac{2 f_0}{T^2} \frac{1}{n^2 \omega_0^2} \left( \sin(n \omega_0 T) - T\, n\, \omega_0 \cdot \cos(n \omega_0 T) \right) =$$

$$= - \frac{2 f_0}{T^2} \cdot \frac{T}{n \omega_0} = - \frac{f_0}{\pi n} \qquad (\omega_0 T = 2\pi)$$

$$\int x \sin(a x)\, dx = \frac{1}{a^2} \left[ \sin(a x) - a x \cos(a x) \right]$$

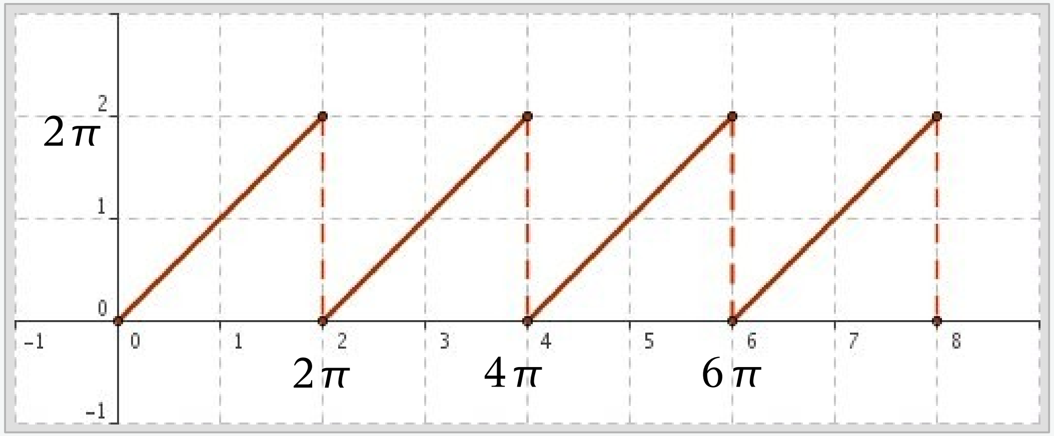
Die Fourier-Reihe der Kippspannung:

$$f(t) = \frac{f_0}{2} - \frac{f_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin(n \omega_0 t) =$$

$$= \frac{f_0}{2} - \frac{f_0}{\pi} \left[ \sin(\omega_0 t) + \frac{1}{2} \sin(2 \omega_0 t) + \frac{1}{3} \sin(3 \omega_0 t) + \ldots \right]$$

# Ein Anwendungsbeispiel: Fourier-Zerlegung einer Kippspannung

$$f(x) = x\,, \quad 0 \leqslant x < 2\pi\,, \quad \omega_0 = \frac{2\pi}{T} = 1, \quad T = 2\pi$$

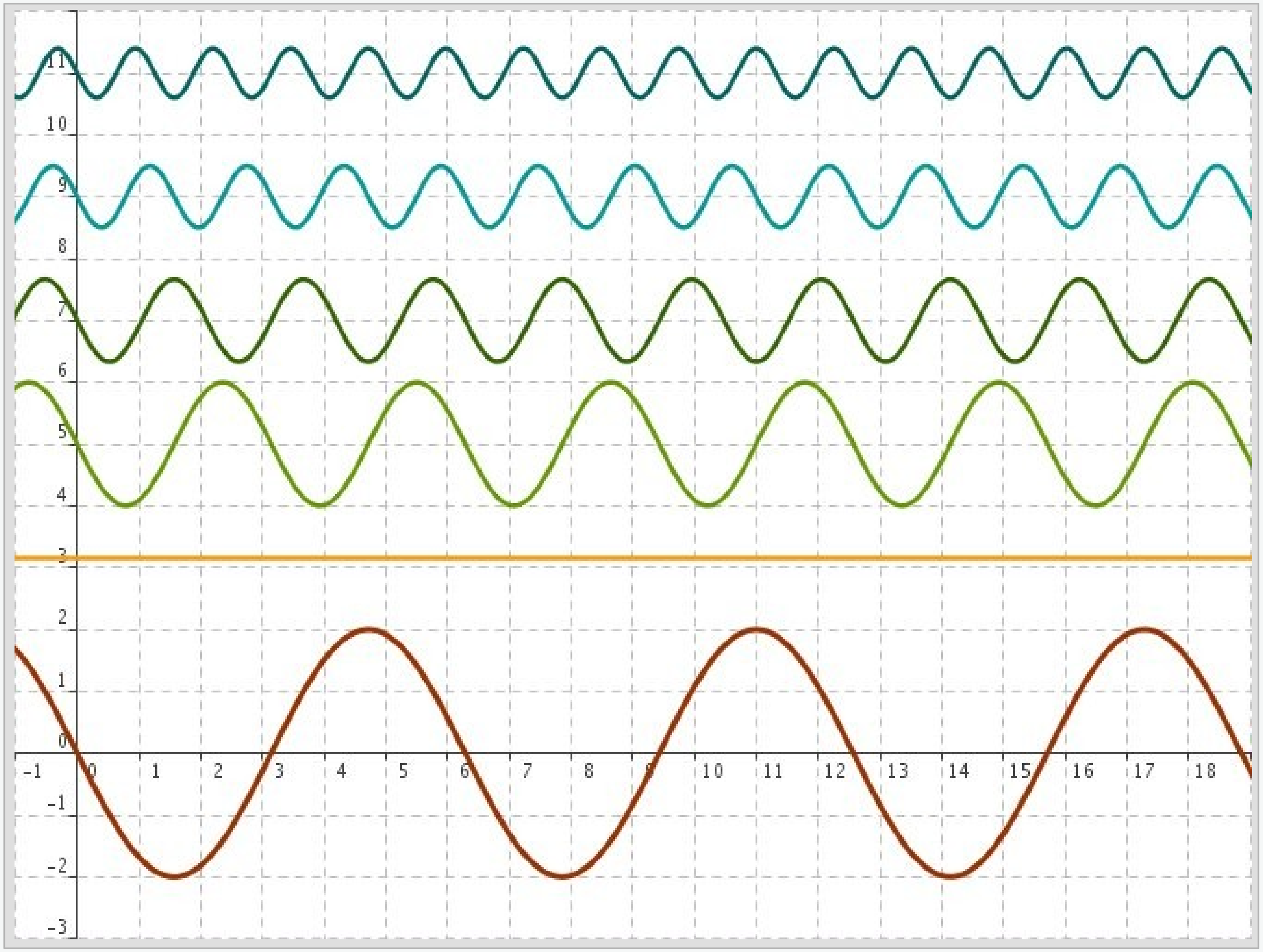



$$a_0 = \frac{2}{T} \cdot \int_0^T f(x)\, dx = \frac{1}{\pi} \cdot \int_0^{2\pi} x\, dx = 2\pi$$

$$a_n = \frac{2}{T} \cdot \int_0^T f(x) \cdot \cos(n\,\omega_0\, x)\, dx = \frac{1}{\pi} \cdot \int_0^{2\pi} x \cos(n\, x)\, dx = 0$$

$$b_n = \frac{2}{T} \cdot \int_0^T f(x) \cdot \sin(n\,\omega_0\, x)\, dx = \frac{1}{\pi} \cdot \int_0^{2\pi} x\, \sin(n\, x)\, dx = -\frac{2}{n}$$

$$\boxed{f(x) = \pi - 2\left(\sin x + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \ldots\right)}$$




$- \dfrac{2}{5} \sin(5x)$

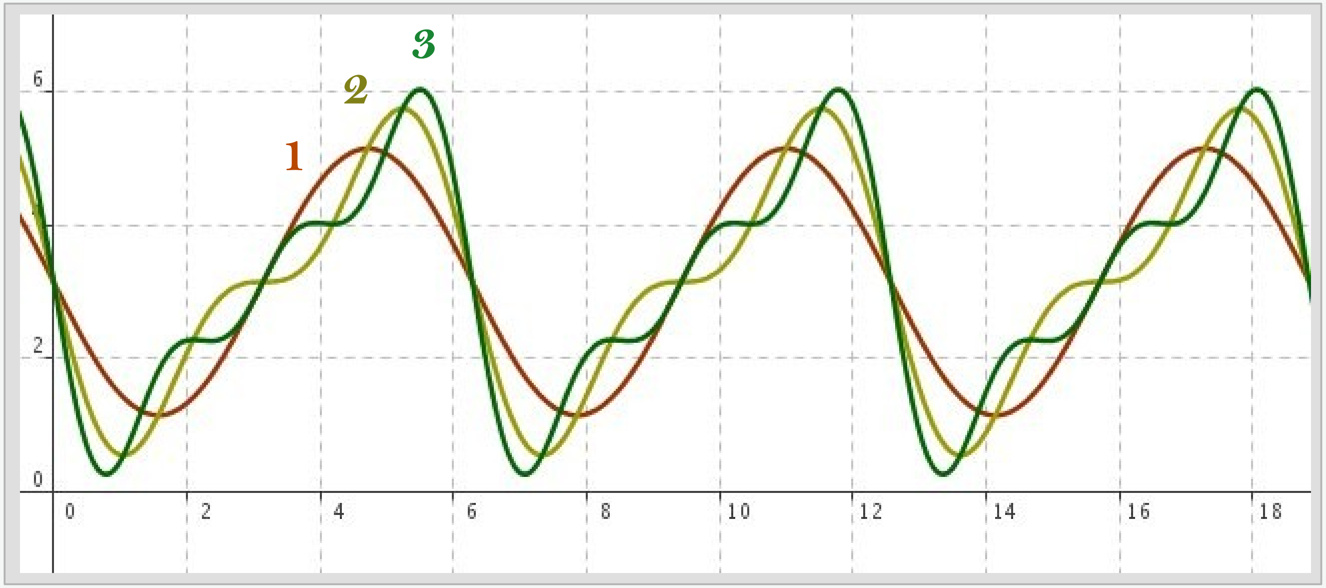
$- \dfrac{1}{2} \sin(4x)$

$- \dfrac{2}{3} \sin(3x)$

$- \sin(2x)$

$\pi$

$- 2 \sin x$

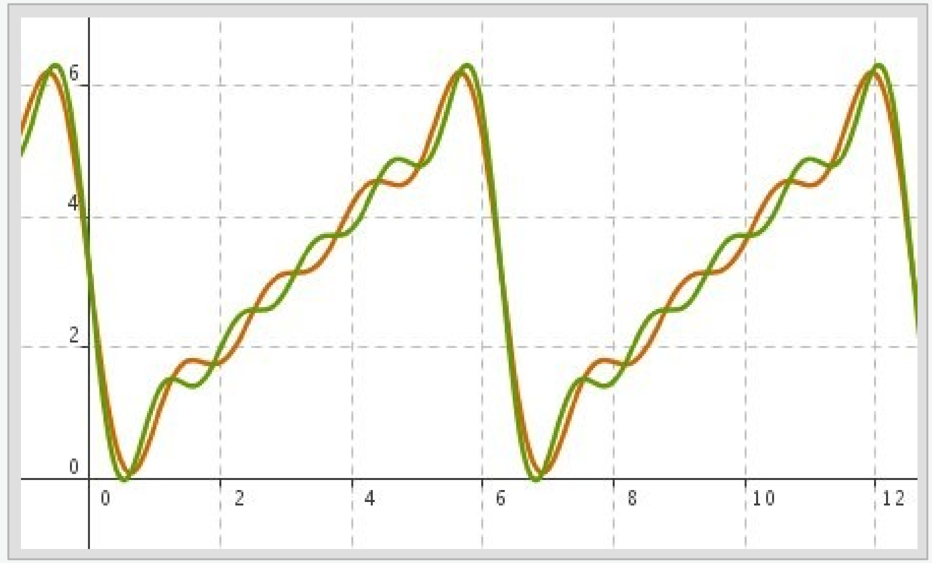



1. Näherung: $f_1(x) = \pi - 2\,\sin x$

2. Näherung: $f_2(x) = \pi - 2\,\sin x - \sin(2\,x)$

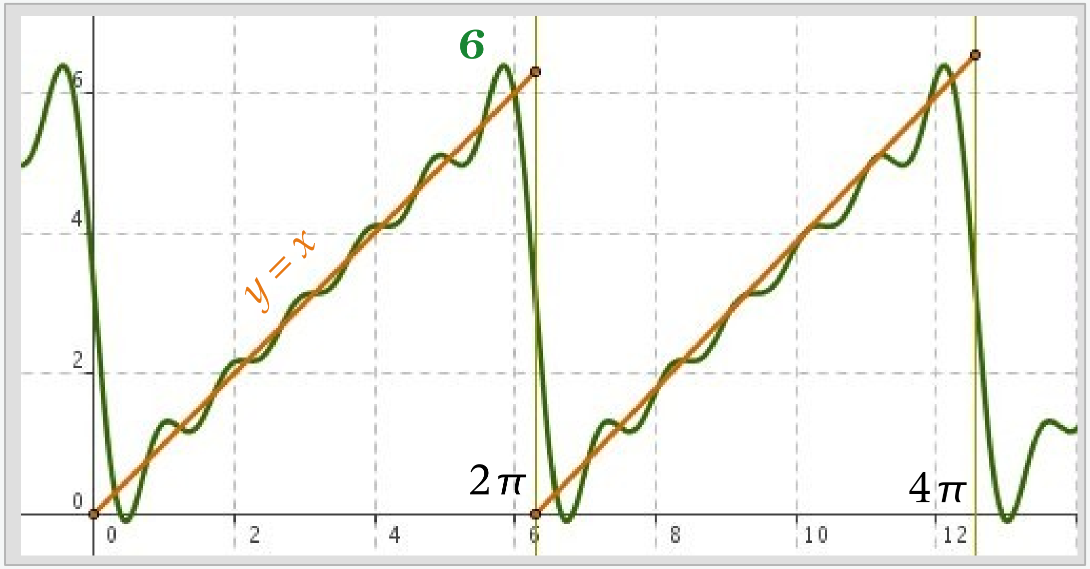
3. Näherung: $f_3(x) = \pi - 2\,\sin x - \sin(2\,x) - \dfrac{2}{3}\,\sin(3\,x)$

# Ein Anwendungsbeispiel: Fourier-Zerlegung einer Kippspannung

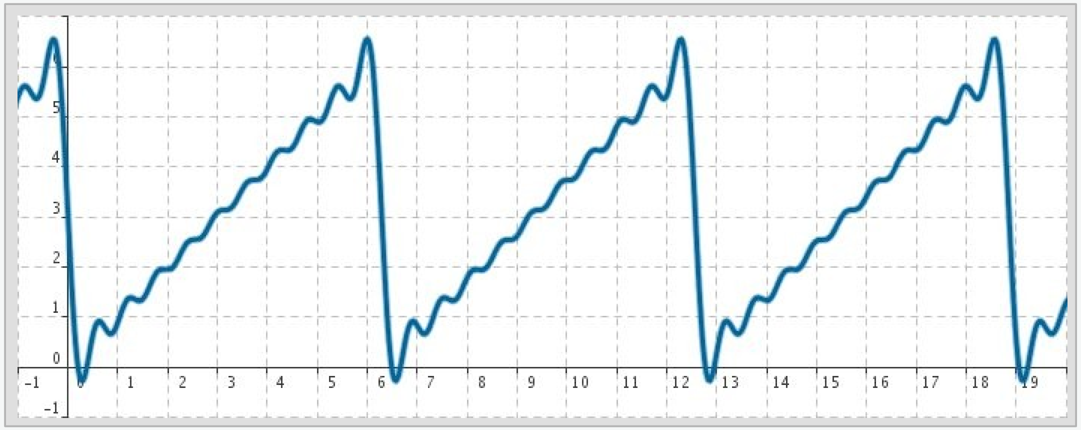



4. Näherung: $$f_4(x) = \pi - 2\sin x - \sin(2x) - \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x)$$

5. Näherung: $$f_5(x) = f_4(x) - \frac{2}{5}\sin(5x)$$

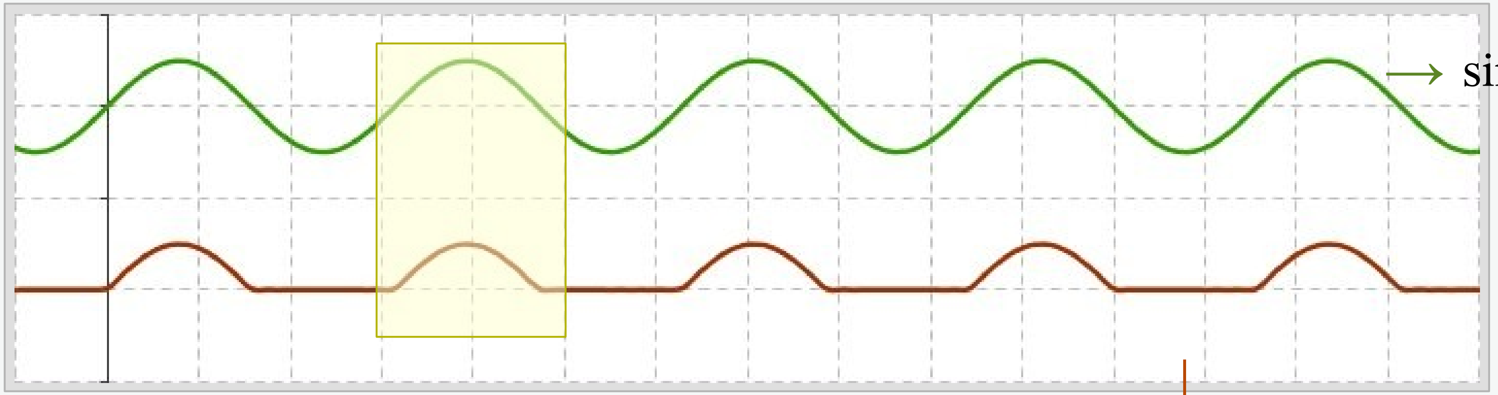



6. Näherung: $f_6(x) = f_5(x) - \dfrac{1}{3}\sin(6x)$

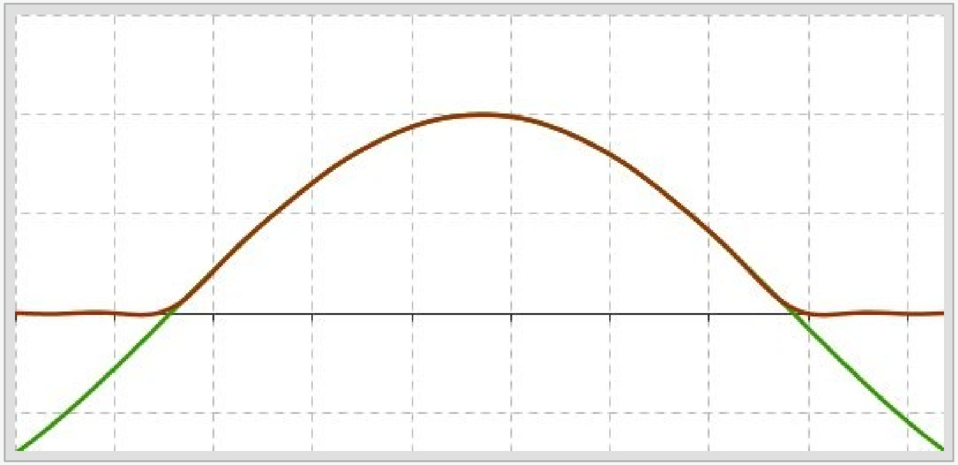


10. Näherung: $f_{10}(x) = \pi - 2\sin x - \ldots - \dfrac{1}{5}\sin(10x)$

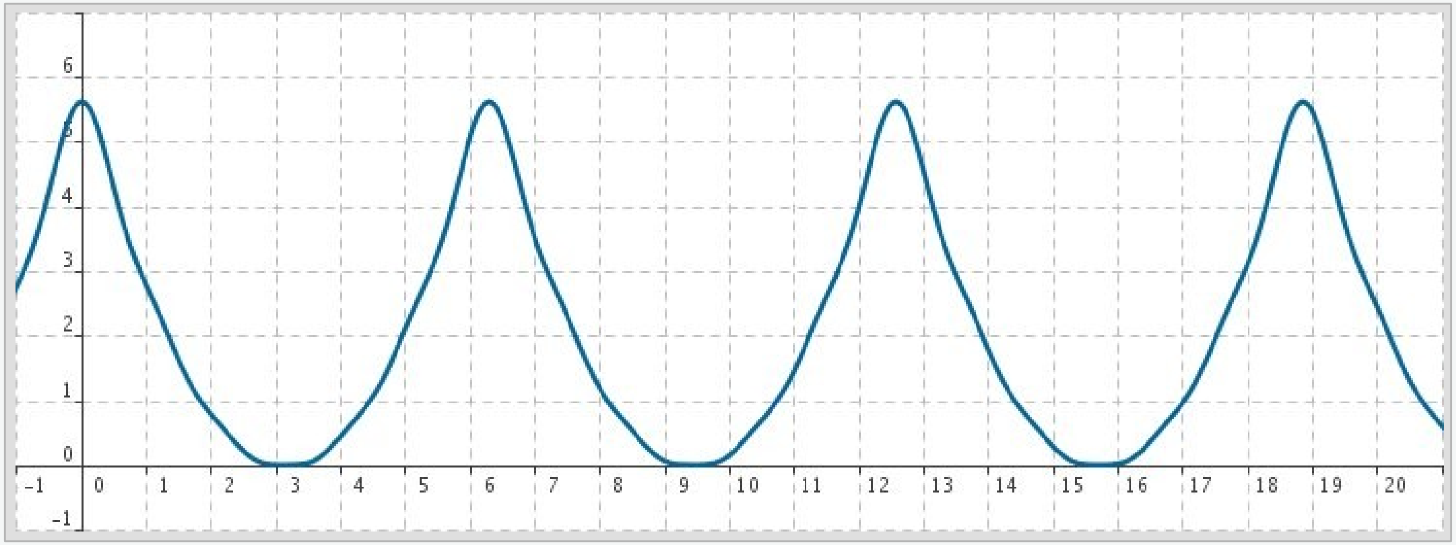
$$f(t) = \begin{cases} \sin(\omega_0 t) & \left(0 \leqslant t < \dfrac{T}{2}\right) \\ 0 & \left(\dfrac{T}{2} \leqslant t < T\right) \end{cases}$$



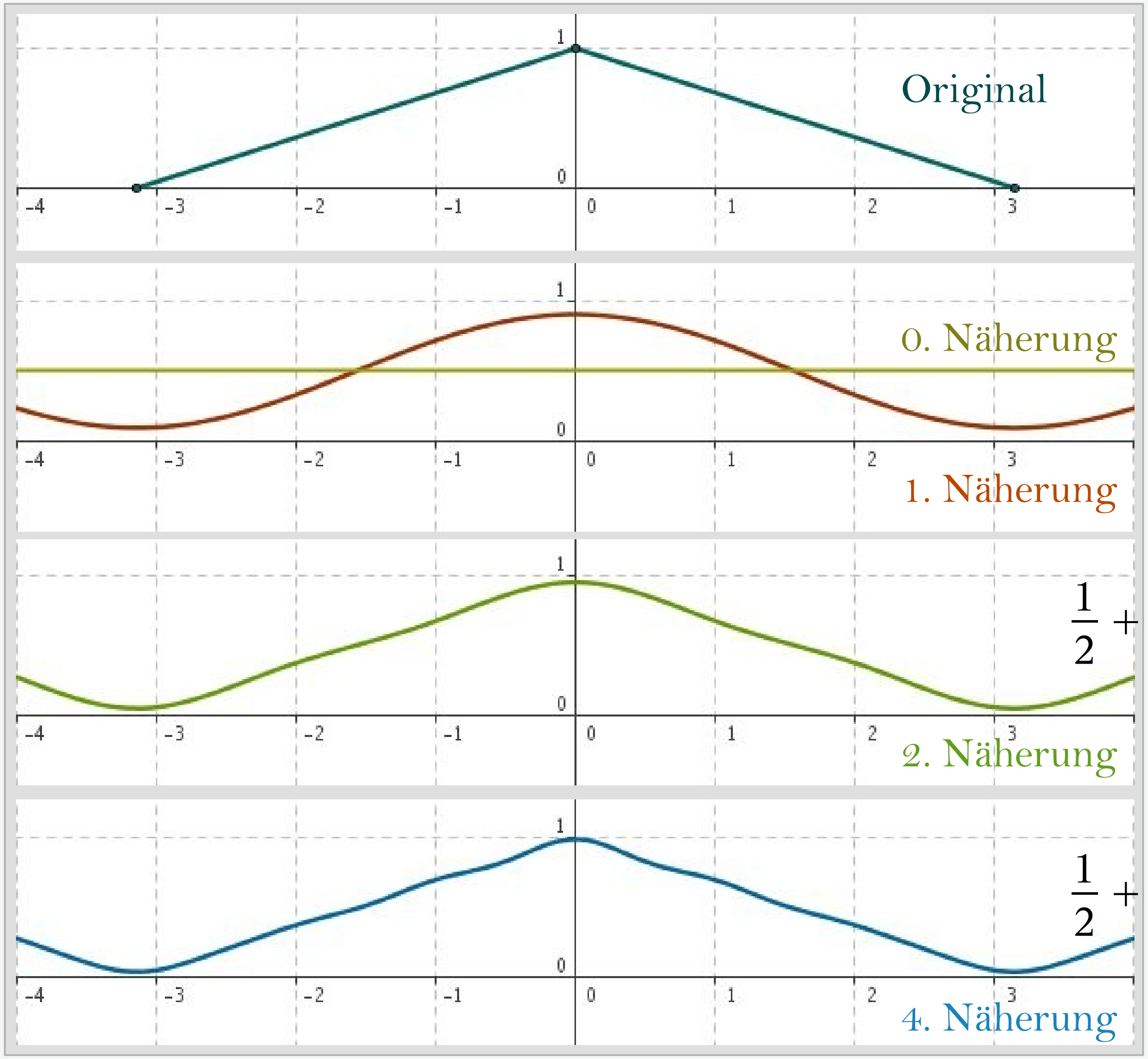

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin(\omega_0 t) - \frac{2}{\pi}\left(\frac{1}{1\cdot 3}\cos(2\omega_0 t) + \frac{1}{3\cdot 5}\cos(4\omega_0 t) + \frac{1}{5\cdot 7}\cos(6\omega_0 t) + \ldots\right)$$



$$f(t) = \frac{4}{T^2}\left(t - \frac{2}{T}\right)^2$$




$$f(t) = \frac{1}{3} + \frac{4}{\pi^2}\left(\cos(\omega_0 t) + \frac{1}{2^2}\cos(2\omega_0 t) + \frac{1}{3^2}\cos(3\omega_0 t) + \ldots\right)$$




$$\frac{1}{2}$$

$$\frac{1}{2} + \frac{4}{\pi^2} \cos(\omega_0 t)$$

$$\frac{1}{2} + \frac{4}{\pi^2} \left( \cos(\omega_0 t) + \frac{1}{3^2} \cos(3 \omega_0 t) \right)$$

$$\frac{1}{2} + \frac{4}{\pi^2} \left( \cos(\omega_0 t) + \ldots \frac{1}{7^2} \cos(7 \omega_0 t) \right)$$

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} \left( \cos(\omega_0 t) + \frac{1}{3^2} \cos(3 \omega_0 t) + \frac{1}{5^2} \cos(5 \omega_0 t) + \ldots \right)$$