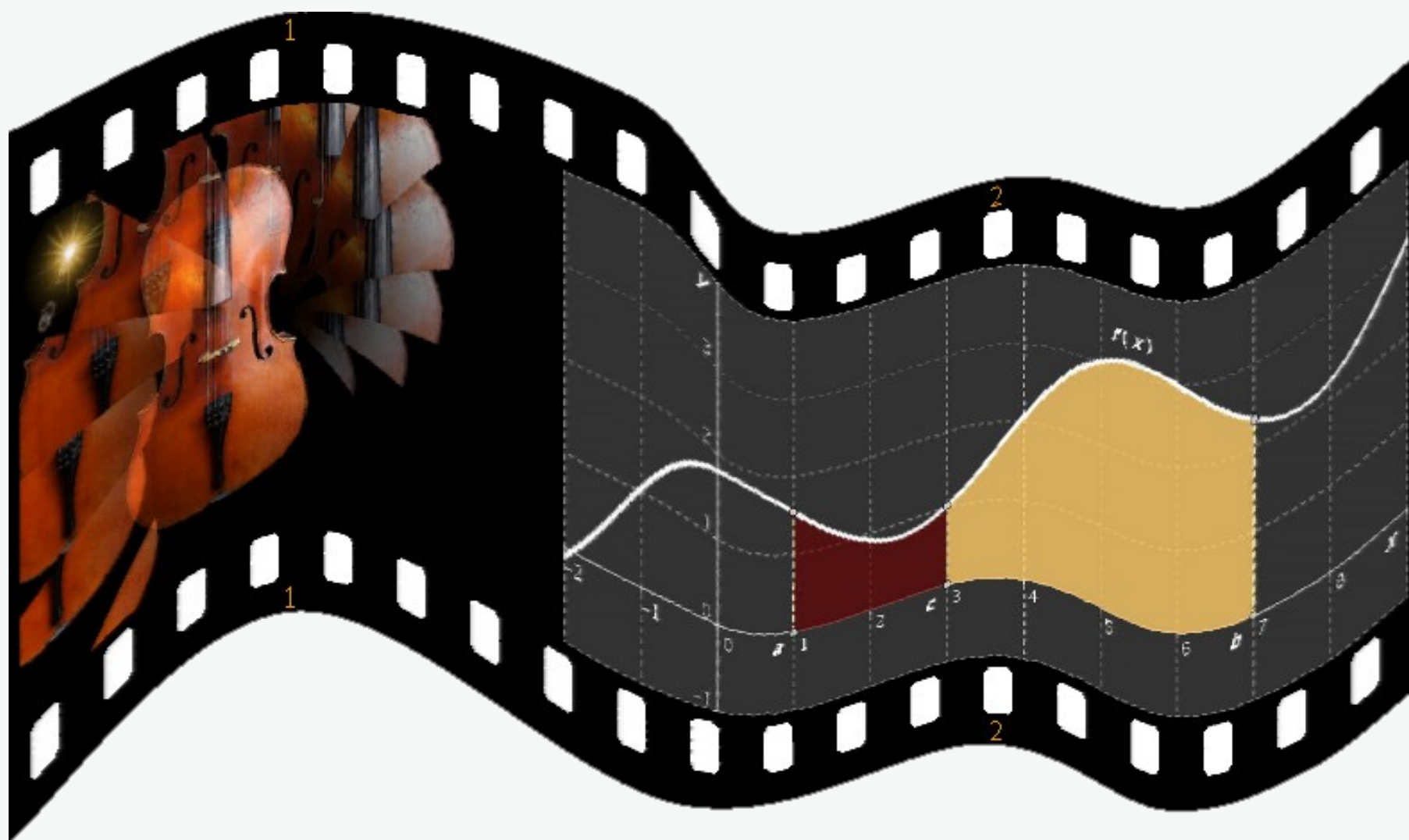




Integrationsregeln, Integration durch Substitution







Faktorregel:

$$\int_a^b C \cdot f(x) \, dx = C \cdot \int_a^b f(x) \, dx$$

Summenregel:

$$\int_a^b (f_1(x) + \dots + f_n(x)) \, dx = \int_a^b f_1(x) \, dx + \dots + \int_a^b f_n(x) \, dx$$

Vertauschungsregel:

$$\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$$

$$\int_a^a f(x) \, dx = 0$$

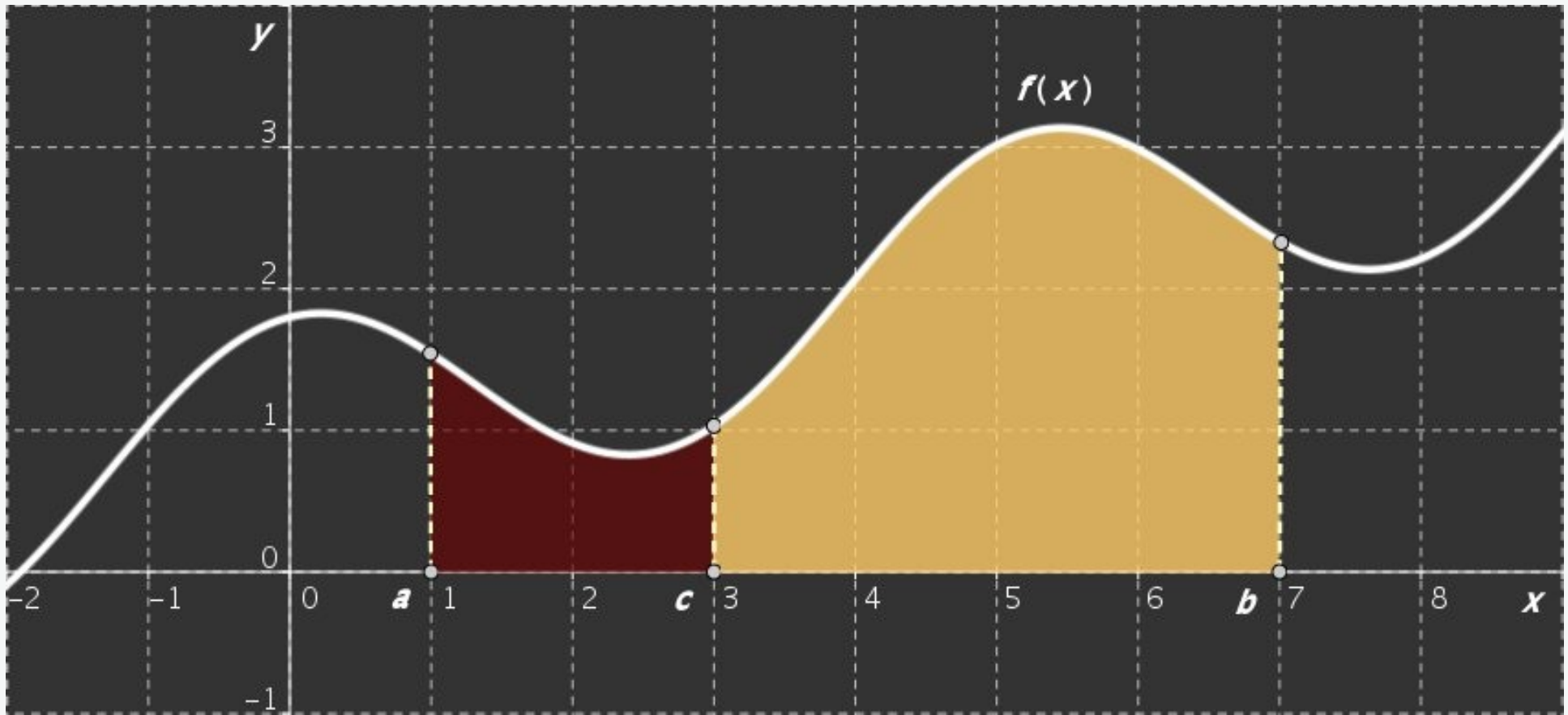
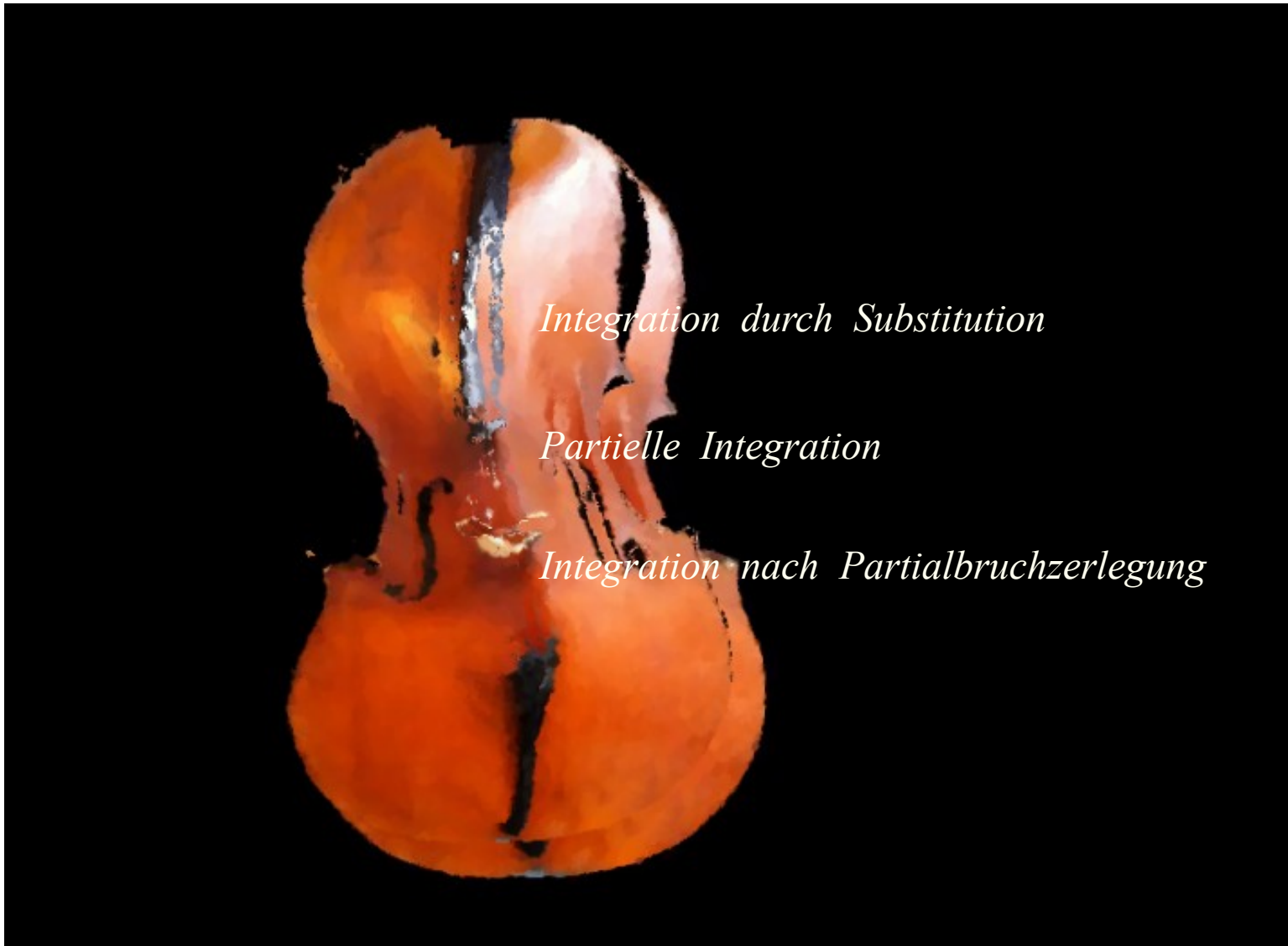


Abb. 1: Intervallzerlegung

Zerlegung des Integrationsintervalls in zwei Teilintervalle:

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \quad (a \leq c \leq b)$$



Integrationsmethoden

Die Integration durch Substitution: Beispiel 1

Manche Integrale, die nicht zu Grundintegralen gehören, lassen sich durch eine geeignete Substitution in Grundintegralen überführen

$$\int x \cdot \cos(x^2) dx$$

Substitution: $u = x^2$

Das “alte” Differential dx ist durch das “neue” Differential du auszudrücken.

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Leftrightarrow x dx = \frac{1}{2} du$$

$$\int x \cdot \cos(x^2) dx = \frac{1}{2} \int \cos u du = \frac{1}{2} \sin u + C = \frac{1}{2} \sin(x^2) + C$$

$$\int x \cdot \cos(x^2) dx = \frac{1}{2} \sin(x^2) + C$$

Ziel der Substitution ist es, den zu integrierenden Ausdruck zu vereinfachen: Der Integrand wird durch eine neue Variable ausgedrückt und umgeformt.



$$\int f(g(x)) g'(x) dx = \int f(u) du, \quad u = g(x), \quad du = g'(x) dx$$

Das Integral eines Produktes lässt sich immer dann berechnen, wenn ein Faktor $f(g(x))$ eine Funktion einer inneren Funktion und der andere Faktor die Ableitung $g'(x)$ der inneren Funktion ist, sofern für die Funktion f unter Beachtung der Substitution $u = g(x)$ das Integral gelöst werden kann.



Der Erfolg einer solchen Substitution ist abhängig von der richtigen Wahl der Substitution $u = g(x)$. Dies setzt gewisse Erfahrungen voraus, die sich nur durch gründliches Üben erwerben lassen. Vor der Festlegung der Substitution verschaffe man sich immer erst Klarheit über die Struktur des Integranden und berücksichtige auch den Einfluß des Differentials dx .

Beispiel 2: $\int \sin x \cos^2 x \, dx$, $u = \cos x$, $u' = -\sin x$

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Leftrightarrow \sin x \, dx = -du$$

$$\int \sin x \cos^2 x \, dx = -\int u^2 \, du = -\frac{u^3}{3} + C = -\frac{1}{3} \cos^3 x + C$$

Beispiel 3: $\int \frac{x \, dx}{\sqrt{1-x^2}}$, $u = 1 - x^2$, $u' = -2x$

$$\int \frac{x \, dx}{\sqrt{1-x^2}} = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-x^2} + C$$

Beispiel 4: $\int \sqrt{4 + 5x} \, dx, \quad u = 4 + 5x, \quad u' = 5$

$$u = 4 + 5x \Rightarrow \frac{du}{dx} = 5 \Leftrightarrow dx = \frac{1}{5} du$$

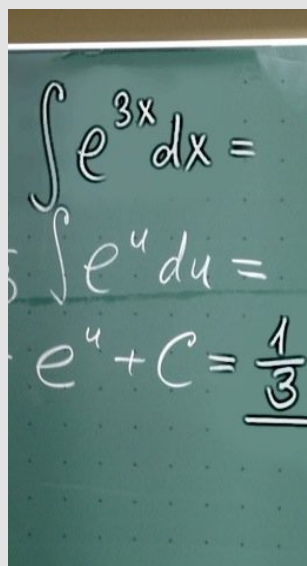
$$\int \sqrt{4 + 5x} \, dx = \int u^{1/2} \frac{du}{5} = \frac{1}{5} \int u^{1/2} \, du = \frac{2}{15} (4 + 5x)^{3/2} + C$$

Beispiel 5: $\int \frac{6x^2}{(1 - 4x^3)^3} \, dx, \quad u = 1 - 4x^3, \quad u' = -12x^2$

$$u = 1 - 4x^3 \Rightarrow \frac{du}{dx} = -12x^2 \Leftrightarrow dx = -\frac{du}{12x^2}$$

$$\int \frac{6x^2}{(1 - 4x^3)^3} \, dx = -\int \frac{6x^2}{u^3} \frac{du}{12x^2} = -\frac{1}{2} \int \frac{du}{u^3} =$$

$$= -\frac{1}{2} \int u^{-3} \, du = \frac{1}{4} u^{-2} + C = \frac{1}{4(1 - 4x^3)^2} + C$$



Handwritten integration steps on a green chalkboard:

$$\int e^{3x} dx =$$
$$\int e^u du =$$
$$e^u + C = \underline{\underline{\frac{1}{3}}}$$

$$I_1 = \int \frac{2x - 3}{x^2 - 3x + 12} dx, \quad u = x^2 - 3x + 12$$

$$I_2 = \int \frac{3x^2 - 5}{x^3 - 5x - 6} dx, \quad u = x^3 - 5x - 6$$

$$I_3 = \int \frac{dx}{x \ln x}, \quad u = \ln x$$

$$I_4 = \int \frac{x dx}{\cos^2(x^2)}, \quad u = x^2$$

$$I_5 = \int \frac{x dx}{\sqrt{1+x}}, \quad u = 1+x$$

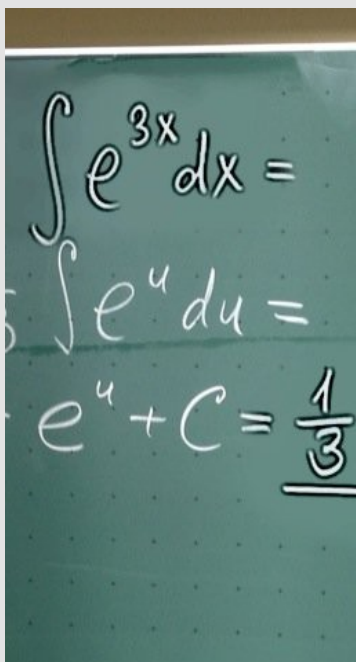
$$I_1 = \int \frac{2x - 3}{x^2 - 3x + 12} dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2 - 3x + 12| + C$$

$$I_2 = \int \frac{3x^2 - 5}{x^3 - 5x - 6} dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^3 - 5x - 6| + C$$

$$I_3 = \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$$

$$I_4 = \int \frac{x dx}{\cos^2(x^2)} = \frac{1}{2} \int \frac{dx}{\cos^2 u} = \frac{1}{2} \tan u + C = \frac{1}{2} \tan x^2 + C$$

$$\begin{aligned} I_5 &= \int \frac{x dx}{\sqrt{1+x}} = \int \frac{(u-1) du}{\sqrt{u}} = \int (u^{1/2} - u^{-1/2}) du = \frac{2}{3} \sqrt{u} (u-3) + C = \\ &= \frac{2}{3} \sqrt{1+x} (x-2) + C \end{aligned}$$



Handwritten integration formulas on a green chalkboard:

$$\int e^{3x} dx =$$
$$\int e^u du =$$
$$e^u + C = \underline{\underline{\frac{1}{3}}}$$

$$I_1 = \int (2x - 7)^3 dx$$

$$I_2 = \int (12 - 2x)^5 dx$$

$$I_3 = \int x (2x^2 + 11)^2 dx$$

$$I_4 = \int x^2 (7 - 3x^3)^5 dx$$

$$I_5 = \int \sqrt{3x - 8} dx$$

$$I_6 = \int \sqrt[3]{(2x + 5)^2} dx$$

$$I_7 = \int x \sqrt{2x^2 - 3} dx$$

$$I_8 = \int (2x - 1) \sqrt{9x^2 - 9x - 4} dx$$

$$I_9 = \int (5x - 12)^{-1/2} dx$$

Die Integration durch Substitution: Lösung 2

$$I_1 = \int (2x - 7)^3 dx = \frac{1}{8} (2x - 7)^4 + C, \quad u = 2x - 7$$

$$I_2 = \int (12 - 2x)^5 dx = -\frac{16}{3} (6 - x)^6 + C, \quad u = 6 - x$$

The image shows a chalkboard with handwritten mathematical work. At the top, there are several integral formulas: $I_2 = \int x \ln x dx$, $I_3 = \int \frac{\ln x}{x} dx$, $I_6 = \int x^x \cos x dx$, and a boxed formula $\int u' v dx = uv - \int u'' v dx$. The main part of the board shows the calculation for $I_2 = \int (12 - 2x)^5 dx$. It starts with the substitution $u = 12 - 2x$, then finds $\frac{du}{dx} = -2$ and $dx = -\frac{du}{2}$. The integral is then transformed into $-\frac{1}{2} \int u^5 du$, which is solved to get $-\frac{1}{2} \cdot \frac{u^6}{6} + C = -\frac{u^6}{12} + C$. This is then substituted back to $-\frac{(12 - 2x)^6}{12} + C$. A simplification is shown below: $12 - 2x = 2(6 - x)$, so $(12 - 2x)^6 = (2 \cdot (6 - x))^6 = 2^6 \cdot (6 - x)^6$. Finally, the coefficient is simplified: $\frac{2^6}{12} = \frac{2^6}{2^2 \cdot 3} = \frac{2^4}{3} = \frac{16}{3}$. The final result is $-\frac{16}{3} (6 - x)^6 + C$, which is underlined in red.

$$\begin{aligned} I_2 &= \int (12 - 2x)^5 dx, \quad u = 12 - 2x, \quad \frac{du}{dx} = -2, \quad dx = -\frac{du}{2} \\ &= -\frac{1}{2} \int u^5 du = -\frac{1}{2} \cdot \frac{u^6}{6} + C = -\frac{u^6}{12} + C = \\ &= -\frac{(12 - 2x)^6}{12} + C = -\frac{2^6}{12} \cdot (6 - x)^6 + C = \underline{-\frac{16}{3} (6 - x)^6 + C} \end{aligned}$$
$$\begin{aligned} 12 - 2x &= 2(6 - x) \\ (12 - 2x)^6 &= (2 \cdot (6 - x))^6 = 2^6 \cdot (6 - x)^6 \\ \frac{2^6}{12} &= \frac{2^6}{2^2 \cdot 3} = \frac{2^4}{3} = \frac{16}{3} \end{aligned}$$

Die Integration durch Substitution: Lösung 2

$$I_3 = \int x (2x^2 + 11)^2 dx = \frac{1}{12} (2x^2 + 11)^3 + C, \quad u = 2x^2 + 11$$

$$I_4 = \int x^2 (7 - 3x^3)^5 dx = -\frac{1}{54} (7 - 3x^3)^6 + C, \quad u = 7 - 3x^3$$

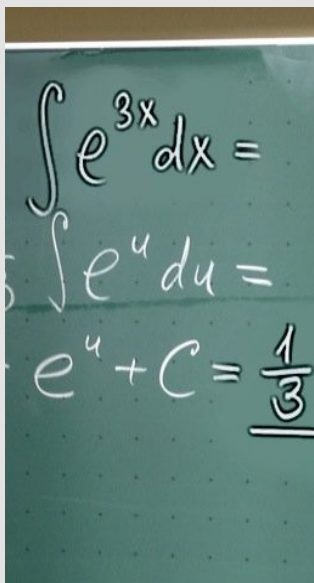
$$\begin{aligned} I_5 &= \int \sqrt{3x - 8} dx = \frac{2}{9} (3x - 8)^{3/2} + C = \\ &= \frac{2}{9} (3x - 8) \sqrt{3x - 8} + C, \quad u = 3x - 8 \end{aligned}$$

$$\begin{aligned} I_6 &= \int \sqrt[3]{(2x + 5)^2} dx = \frac{3}{10} (2x + 5)^{5/3} + C = \\ &= \frac{3}{10} (2x + 5) \sqrt[3]{(2x + 5)^2} + C, \quad u = 2x + 5 \end{aligned}$$

$$I_7 = \int x \sqrt{2x^2 - 3} dx = \frac{1}{6} (2x^2 - 3)^{3/2} + C, \quad u = 2x^2 - 3$$

$$\begin{aligned} I_8 &= \int (2x - 1) \sqrt{9x^2 - 9x - 4} dx = \frac{2}{27} (9x^2 - 9x - 4)^{3/2} + C \\ &u = 9x^2 - 9x - 4 \end{aligned}$$

$$I_9 = \int (5x - 12)^{-1/2} dx = \frac{2}{5} \sqrt{5x - 12} + C, \quad u = 5x - 12$$



Handwritten integration formulas on a green chalkboard:

$$\int e^{3x} dx =$$
$$\int e^u du =$$
$$e^u + C = \underline{\underline{\frac{1}{3}}}$$

Berechnen Sie folgende unbestimmte Integrale

$$I_{1a} = \int \cos(3x) dx, \quad I_{1b} = \int \cos(ax) dx$$

$$I_{2a} = \int \cos(3x - 2) dx, \quad I_{2b} = \int x \cos(3x^2 - 2) dx$$

$$I_{3a} = \int \cos^2 x dx, \quad I_{3b} = \int \sin^2 x dx$$

$$I_{4a} = \int \frac{\cos x}{1 + 2 \sin x} dx, \quad I_{4b} = \int \frac{\sin x}{1 + 3 \cos x} dx$$

$$I_{5a} = \int \frac{\cos(3x)}{5 - 6 \sin(3x)} dx, \quad I_{5b} = \int \frac{\sin(4x)}{11 + 2 \cos(4x)} dx$$

$$I_{6a} = \int \sin^3 x \cos x dx, \quad I_{6b} = \int \cos^4 x \sin x dx$$

$$I_{1a} = \int \cos(3x) dx = \frac{1}{3} \sin(3x) + C$$

$$I_{1b} = \int \cos(ax) dx = \frac{1}{a} \sin(ax) + C$$

$$I_{2a} = \int \cos(3x - 2) dx = \frac{1}{3} \sin(3x - 2) + C$$

$$I_{2b} = \int x \cos(3x^2 - 2) dx = \frac{1}{6} \sin(3x^2 - 2) + C$$

$$u = 3x^2 - 2, \quad x dx = -\frac{du}{6}$$

$$\begin{aligned} I_{3a} &= \int \cos^2 x dx = \frac{1}{2} \int (\cos(2x) + 1) dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos(2x) dx = \\ &= \frac{1}{2} \int dx + \frac{1}{4} \int \cos u du = \frac{x}{2} + \frac{\sin(2x)}{4} + C \quad (u = 2x) \end{aligned}$$

$$\begin{aligned} I_{3b} &= \int \sin^2 x dx = \frac{1}{2} \int (1 - \cos(2x)) dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos(2x) dx = \\ &= \frac{x}{2} - \frac{\sin(2x)}{4} + C = \frac{x}{2} - \frac{1}{2} \sin x \cos x + C \end{aligned}$$

$$I_{4a} = \int \frac{\cos x}{1 + 2 \sin x} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |1 + 2 \sin x| + C$$
$$u = 1 + 2 \sin x, \quad \cos x dx = \frac{du}{2}$$

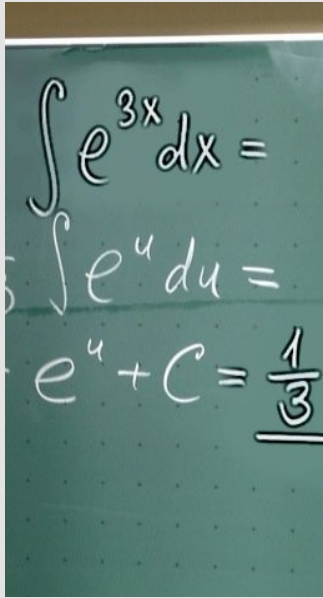
$$I_{4b} = \int \frac{\sin x}{1 + 3 \cos x} dx = -\frac{1}{3} \ln |1 + 3 \cos x| + C$$

$$I_{5a} = \int \frac{\cos(3x)}{5 - 6 \sin(3x)} dx = -\frac{1}{18} \ln |5 - 6 \sin(3x)| + C$$
$$u = 5 - 6 \sin(3x), \quad \cos(3x) dx = -\frac{du}{18}$$

$$I_{5b} = \int \frac{\sin(4x)}{11 + 2 \cos(4x)} dx = -\frac{1}{8} \ln |11 + 2 \cos(4x)| + C$$

$$I_{6a} = \int \sin^3 x \cos x dx = \frac{1}{4} \sin^4 x + C$$

$$I_{6b} = \int \cos^4 x \sin x dx = -\frac{1}{5} \cos^5 x + C$$



Handwritten mathematical work on a green chalkboard:

$$\int e^{3x} dx =$$
$$\int e^u du =$$
$$e^u + C = \underline{\underline{\frac{1}{3}}}$$

Berechnen Sie folgende unbestimmte Integrale

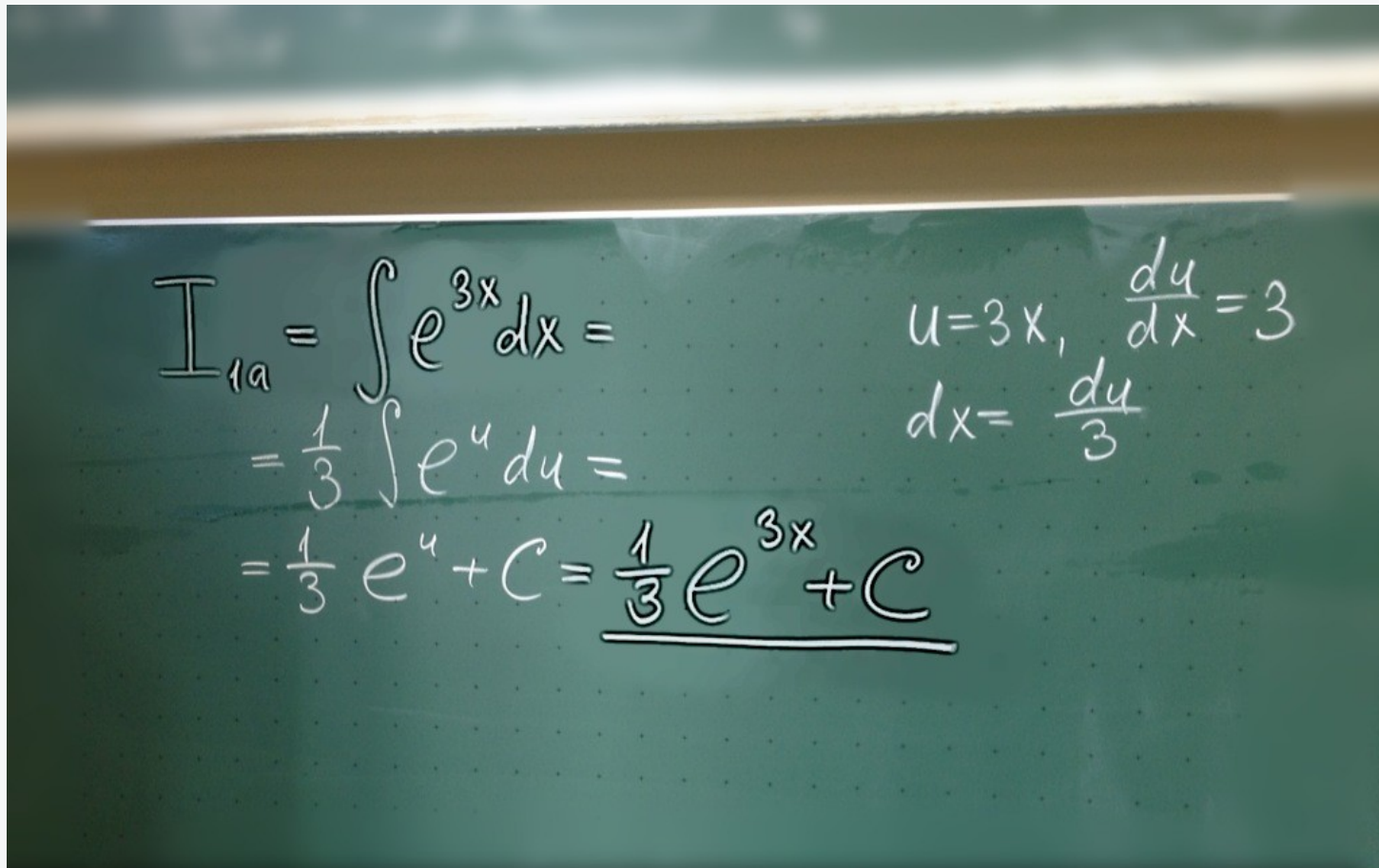
$$I_{1a} = \int e^{3x} dx, \quad I_{1b} = \int e^{3x-4} dx$$

$$I_{2a} = \int x e^{2x^2} dx, \quad I_{2b} = \int x^3 e^{5x^4-1} dx$$

Die Integration durch Substitution: Lösung 4

$$I_{1a} = \int e^{3x} dx = \frac{1}{3} e^{3x} + C, \quad u = 3x$$

$$I_{1b} = \int e^{3x-4} dx = \frac{1}{3} e^{3x-4} + C, \quad u = 3x - 4$$



Handwritten solution for the integral I_{1a} using substitution:

$$\begin{aligned} I_{1a} &= \int e^{3x} dx = \\ &= \frac{1}{3} \int e^u du = \\ &= \frac{1}{3} e^u + C = \underline{\underline{\frac{1}{3} e^{3x} + C}} \end{aligned}$$

Substitution: $u = 3x, \frac{du}{dx} = 3$
 $dx = \frac{du}{3}$

Die Integration durch Substitution: Lösung 4

$$I_{2a} = \int x e^{2x^2} dx = \frac{1}{4} e^{2x^2} + C, \quad u = 2x^2$$

$$I_{2b} = \int x^3 e^{5x^4-1} dx = \frac{1}{20} e^{5x^4-1} + C, \quad u = 5x^4 - 1$$

Handwritten solution for I_{2a} on a green chalkboard:

$$\begin{aligned} I_{2a} &= \int x \cdot e^{2x^2} dx = \\ &= \int \cancel{x} \cdot e^u \cdot \frac{du}{4\cancel{x}} = \\ &= \frac{1}{4} \int e^u du = \underline{\underline{\frac{1}{4} e^{2x^2} + C}} \end{aligned}$$

Substitution steps shown on the right:

$$\begin{aligned} u &= 2x^2, & \frac{du}{dx} &= 4x \\ dx &= \frac{du}{4x} \end{aligned}$$

