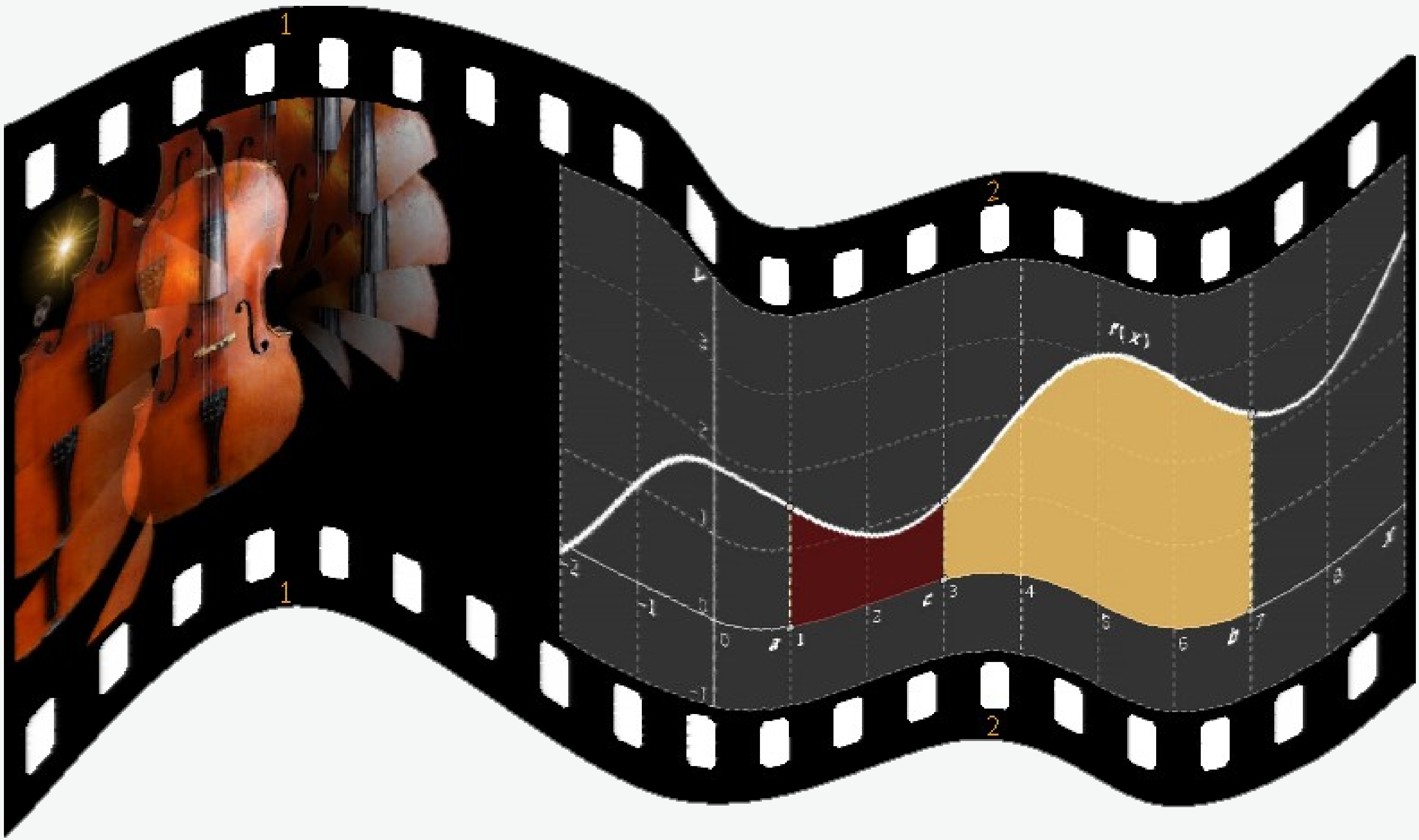
Integrationsregeln, Integration durch Substitution


1
2
f(x)
a
c
b
1
2


3
3
4
4

Faktorregel:

$$\int_a^b C \cdot f(x)\, dx = C \cdot \int_a^b f(x)\, dx$$

Summenregel:

$$\int_a^b (f_1(x) + \ldots + f_n(x))\, dx = \int_a^b f_1(x)\, dx + \ldots + \int_a^b f_n(x)\, dx$$

Vertauschungsregel:

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$

$$\int_a^a f(x)\, dx = 0$$

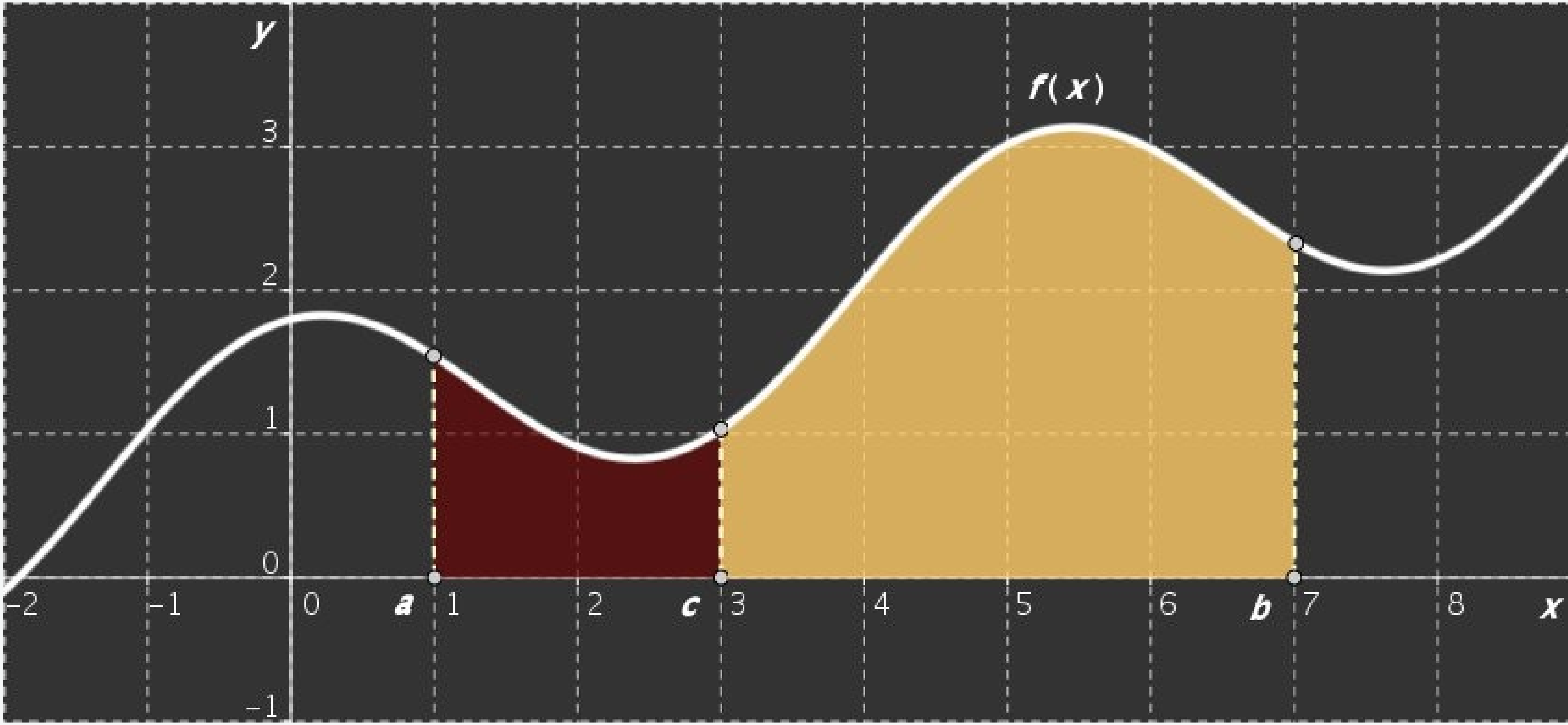


Abb. 1: Intervallzerlegung

Zerlegung des Integrationsintervalls in zwei Teilintervalle:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad (a \leqslant c \leqslant b)$$

Integrationsmethoden

Die Integration durch Substitution: Beispiel 1

Manche Integrale, die nicht zu Grundintegralen gehören, lassen sich durch eine geeignete Substitution in Grundintegralen überführen

$$\int x \cdot \cos\left(x^2\right) \, dx$$

Substitution: $\quad u = x^2$

Das "alte" Differential dx ist durch das "neue" Differential du auszudrücken.

$$u = x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x} \quad \Leftrightarrow \quad x \, dx = \frac{1}{2} \, du$$

$$\int x \cdot \cos\left(x^2\right) dx = \frac{1}{2} \int \cos u \, du = \frac{1}{2} \sin u + C = \frac{1}{2} \sin\left(x^2\right) + C$$

$$\boxed{\int x \cdot \cos\left(x^2\right) dx = \frac{1}{2} \sin\left(x^2\right) + C}$$

Ziel der Substitution ist es, den zu integrierenden Ausdruck zu vereinfachen: Der Integrand wird durch eine neue Variable ausgedrückt und umgeformt.

$$\int f\,(g\,(x))\;g\,'(x)\;dx = \int f\,(u)\;du\,, \qquad u = g\,(x)\,, \qquad du = g\,'(x)\;dx$$

Das Integral eines Produktes lässt sich immer dann berechnen, wenn ein Faktor $f\,(g\,(x))$ eine Funktion einer inneren Funktion und der andere Faktor die Ableitung $g'\,(x)$ der inneren Funktion ist, sofern für die Funktion f unter Beachtung der Substitution $u = g\,(x)$ das Integral gelöst werden kann.

Der Erfolg einer solchen Substitution ist abhängig von der richtigen Wahl der Substitution $u = g(x)$. Dies setzt gewisse Erfahrungen voraus, die sich nur durch gründliches Üben erwerben lassen. Vor der Festlegung der Substitution verschaffe man ich immer erst Klarheit über die Struktur des Integranden und berücksichtige auch den Einfluß des Differentials dx .

Beispiel 2: $\int \sin x \cos^2 x \, dx$, $u = \cos x$, $u' = -\sin x$

$$u = \cos x \quad \Rightarrow \quad \frac{du}{dx} = -\sin x \quad \Leftrightarrow \quad \sin x \, dx = -du$$

$$\int \sin x \cos^2 x \, dx = -\int u^2 \, du = -\frac{u^3}{3} + C = -\frac{1}{3} \cos^3 x + C$$

Beispiel 3: $\int \frac{x \, dx}{\sqrt{1 - x^2}}$, $u = 1 - x^2$, $u' = -2x$

$$\int \frac{x \, dx}{\sqrt{1 - x^2}} = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1 - x^2} + C$$

Beispiel 4: $\int \sqrt{4+5x}\, dx\,, \qquad u = 4+5x\,, \qquad u' = 5$

$$u = 4+5x \quad \Rightarrow \quad \frac{du}{dx} = 5 \quad \Leftrightarrow \quad dx = \frac{1}{5}\, du$$

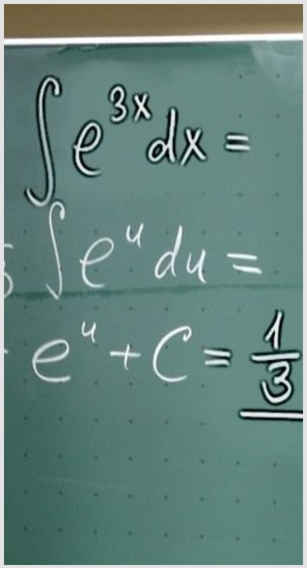
$$\int \sqrt{4+5x}\, dx = \int u^{1/2}\, \frac{du}{5} = \frac{1}{5} \int u^{1/2}\, du = \frac{2}{15}\,(4+5x)^{3/2} + C$$

Beispiel 5: $\int \frac{6x^2}{(1-4x^3)^3}\, dx\,, \qquad u = 1-4x^3\,, \qquad u' = -12x^2$

$$u = 1-4x^3 \quad \Rightarrow \quad \frac{du}{dx} = -12x^2 \quad \Leftrightarrow \quad dx = -\frac{du}{12x^2}$$

$$\int \frac{6x^2}{(1-4x^3)^3}\, dx = -\int \frac{6x^2}{u^3}\, \frac{du}{12x^2} = -\frac{1}{2} \int \frac{du}{u^3} =$$

$$= -\frac{1}{2} \int u^{-3}\, du = \frac{1}{4}\, u^{-2} + C = \frac{1}{4\,(1-4x^3)^2} + C$$



$$I_1 = \int \frac{2x - 3}{x^2 - 3x + 12}\, dx, \qquad u = x^2 - 3x + 12$$

$$I_2 = \int \frac{3x^2 - 5}{x^3 - 5x - 6}\, dx, \qquad u = x^3 - 5x - 6$$

$$I_3 = \int \frac{dx}{x \ln x}, \qquad u = \ln x$$

$$I_4 = \int \frac{x\, dx}{\cos^2(x^2)}, \qquad u = x^2$$

$$I_5 = \int \frac{x\, dx}{\sqrt{1 + x}}, \qquad u = 1 + x$$

$$I_1 = \int \frac{2x - 3}{x^2 - 3x + 12}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2 - 3x + 12| + C$$

$$I_2 = \int \frac{3x^2 - 5}{x^3 - 5x - 6}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^3 - 5x - 6| + C$$

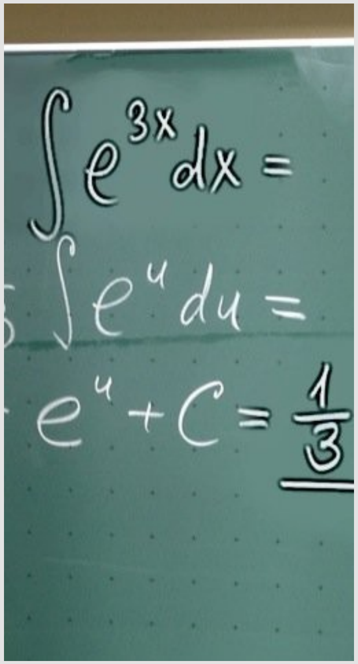
$$I_3 = \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$$

$$I_4 = \int \frac{x\, dx}{\cos^2(x^2)} = \frac{1}{2} \int \frac{dx}{\cos^2 u} = \frac{1}{2} \tan u + C = \frac{1}{2} \tan x^2 + C$$

$$I_5 = \int \frac{x\, dx}{\sqrt{1 + x}} = \int \frac{(u - 1)\, du}{\sqrt{u}} = \int \left(u^{1/2} - u^{-1/2}\right) du = \frac{2}{3} \sqrt{u}\,(u - 3) + C =$$

$$= \frac{2}{3} \sqrt{1 + x}\,(x - 2) + C$$

Die Integration durch Substitution: Aufgabe 2



$$I_1 = \int (2x - 7)^3 \, dx$$

$$I_2 = \int (12 - 2x)^5 \, dx$$

$$I_3 = \int x \, (2x^2 + 11)^2 \, dx$$

$$I_4 = \int x^2 \, (7 - 3x^3)^5 \, dx$$

$$I_5 = \int \sqrt{3x - 8} \, dx$$

$$I_6 = \int \sqrt[3]{(2x + 5)^2} \, dx$$

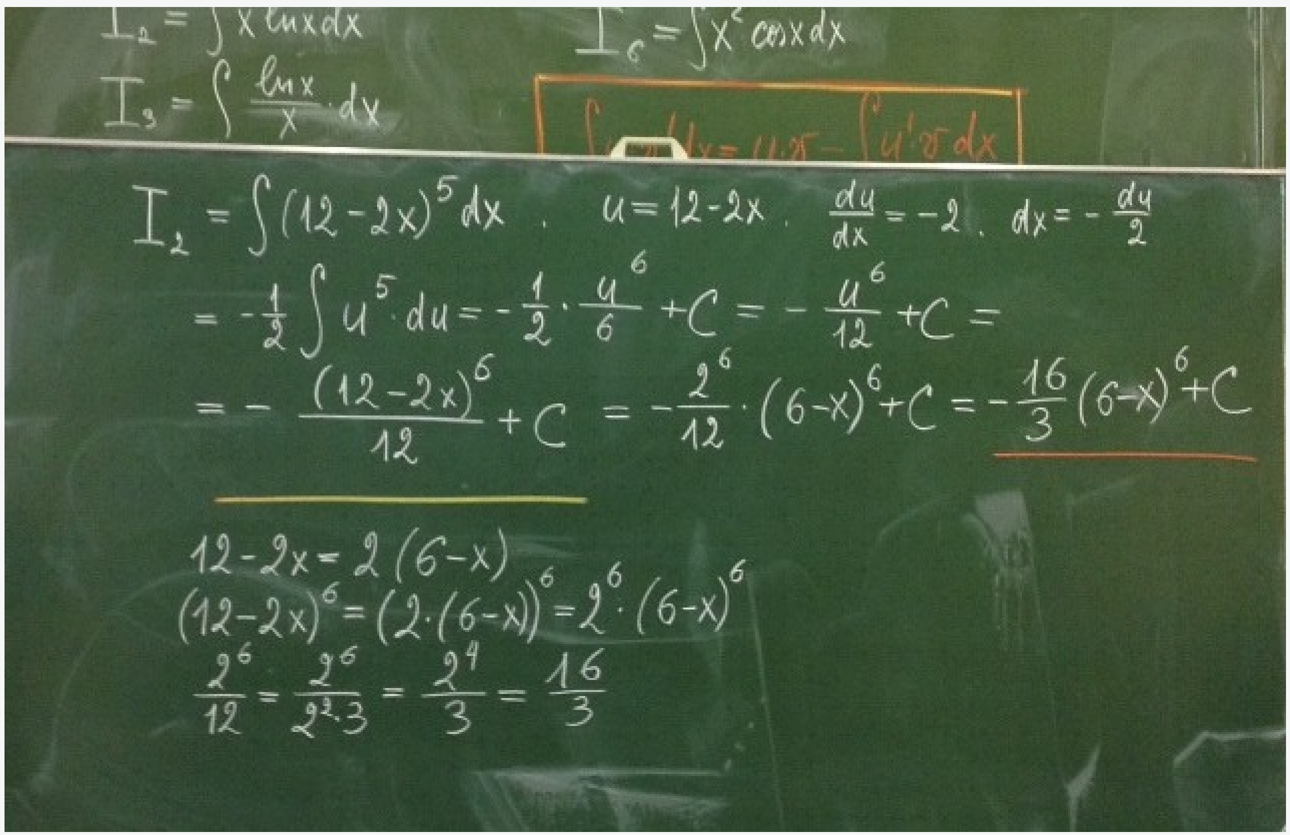
$$I_7 = \int x \sqrt{2x^2 - 3} \, dx$$

$$I_8 = \int (2x - 1) \sqrt{9x^2 - 9x - 4} \, dx$$

$$I_9 = \int (5x - 12)^{-1/2} \, dx$$

$$I_1 = \int (2x - 7)^3 \, dx = \frac{1}{8}\,(2x - 7)^4 + C, \qquad u = 2x - 7$$

$$I_2 = \int (12 - 2x)^5 \, dx = -\frac{16}{3}\,(6 - x)^6 + C, \qquad u = 6 - x$$



Die Integration durch Substitution: Lösung 2

$$I_3 = \int x\ (2x^2 + 11)^2\, dx = \frac{1}{12}\,(2x^2 + 11)^3 + C\,, \qquad u = 2x^2 + 11$$

$$I_4 = \int x^2\ (7 - 3x^3)^5\, dx = -\frac{1}{54}\,(7 - 3x^3)^6 + C\,, \qquad u = 7 - 3x^3$$

$$I_5 = \int \sqrt{3x - 8}\ dx = \frac{2}{9}\,(3x - 8)^{3/2} + C =$$

$$= \frac{2}{9}\,(3x - 8)\,\sqrt{3x - 8} + C\,, \qquad u = 3x - 8$$

$$I_6 = \int \sqrt[3]{(2x + 5)^2}\ dx = \frac{3}{10}\,(2x + 5)^{5/3} + C =$$

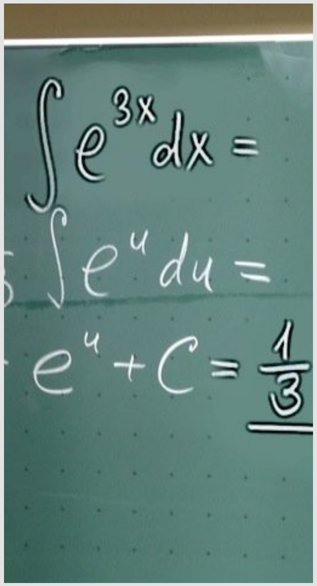
$$= \frac{3}{10}\,(2x + 5)\,\sqrt[3]{(2x + 5)^2} + C\,, \qquad u = 2x + 5$$

$$I_7 = \int x\,\sqrt{2x^2 - 3}\ dx = \frac{1}{6}\,(2x^2 - 3)^{3/2} + C\,, \qquad u = 2x^2 - 3$$

$$I_8 = \int (2x - 1)\,\sqrt{9x^2 - 9x - 4}\ dx = \frac{2}{27}\,(9x^2 - 9x - 4)^{3/2} + C$$

$$u = 9x^2 - 9x - 4$$

$$I_9 = \int (5x - 12)^{-1/2} \, dx = \frac{2}{5} \sqrt{5x - 12} + C, \qquad u = 5x - 12$$



Berechnen Sie folgende unbestimmte Integrale

$$I_{1a} = \int \cos(3x)\, dx, \qquad I_{1b} = \int \cos(ax)\, dx$$

$$I_{2a} = \int \cos(3x - 2)\, dx, \qquad I_{2b} = \int x \cos(3x^2 - 2)\, dx$$

$$I_{3a} = \int \cos^2 x\, dx, \qquad I_{3b} = \int \sin^2 x\, dx$$

$$I_{4a} = \int \frac{\cos x}{1 + 2\sin x}\, dx, \qquad I_{4b} = \int \frac{\sin x}{1 + 3\cos x}\, dx$$

$$I_{5a} = \int \frac{\cos(3x)}{5 - 6\sin(3x)}\, dx, \qquad I_{5b} = \int \frac{\sin(4x)}{11 + 2\cos(4x)}\, dx$$

$$I_{6a} = \int \sin^3 x \cos x\, dx, \qquad I_{6b} = \int \cos^4 x \sin x\, dx$$

$$I_{1a} = \int \cos(3x)\, dx = \frac{1}{3} \sin(3x) + C$$

$$I_{1b} = \int \cos(ax)\, dx = \frac{1}{a} \sin(ax) + C$$

$$I_{2a} = \int \cos(3x - 2)\, dx = \frac{1}{3} \sin(3x - 2) + C$$

$$I_{2b} = \int x \cos(3x^2 - 2)\, dx = \frac{1}{6} \sin(3x^2 - 2) + C$$

$$u = 3x^2 - 2, \qquad x\, dx = -\frac{du}{6}$$

$$I_{3a} = \int \cos^2 x\, dx = \frac{1}{2} \int (\cos(2x) + 1)\, dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos(2x)\, dx =$$

$$= \frac{1}{2} \int dx + \frac{1}{4} \int \cos u\, du = \frac{x}{2} + \frac{\sin(2x)}{4} + C \qquad (u = 2x)$$

$$I_{3b} = \int \sin^2 x\, dx = \frac{1}{2} \int (1 - \cos(2x))\, dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos(2x)\, dx =$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} + C = = \frac{x}{2} - \frac{1}{2} \sin x \cos x + C$$

$$I_{4a} = \int \frac{\cos x}{1 + 2\sin x}\, d\,x = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|\,u\,| + C = \frac{1}{2}\ln|\,1 + 2\sin x\,| + C$$

$$u = 1 + 2\sin x\,, \qquad \cos x\, dx = \frac{du}{2}$$

$$I_{4b} = \int \frac{\sin x}{1 + 3\cos x}\, d\,x = -\frac{1}{3}\ln|\,1 + 3\cos x\,| + C$$

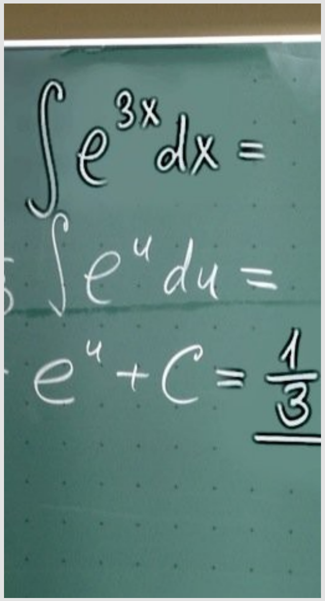
$$I_{5a} = \int \frac{\cos(3\,x)}{5 - 6\sin(3\,x)}\, d\,x = -\frac{1}{18}\ln|\,5 - 6\sin(3\,x)\,| + C$$

$$u = 5 - 6\sin(3\,x)\,, \qquad \cos(3\,x)\, dx = -\frac{du}{18}$$

$$I_{5b} = \int \frac{\sin(4\,x)}{11 + 2\cos(4\,x)}\, d\,x = -\frac{1}{8}\ln|\,11 + 2\cos(4\,x)\,| + C$$

$$I_{6a} = \int \sin^3 x \cos x\; d\,x = \frac{1}{4}\sin^4 x + C$$

$$I_{6b} = \int \cos^4 x \sin x\; d\,x = -\frac{1}{5}\cos^5 x + C$$



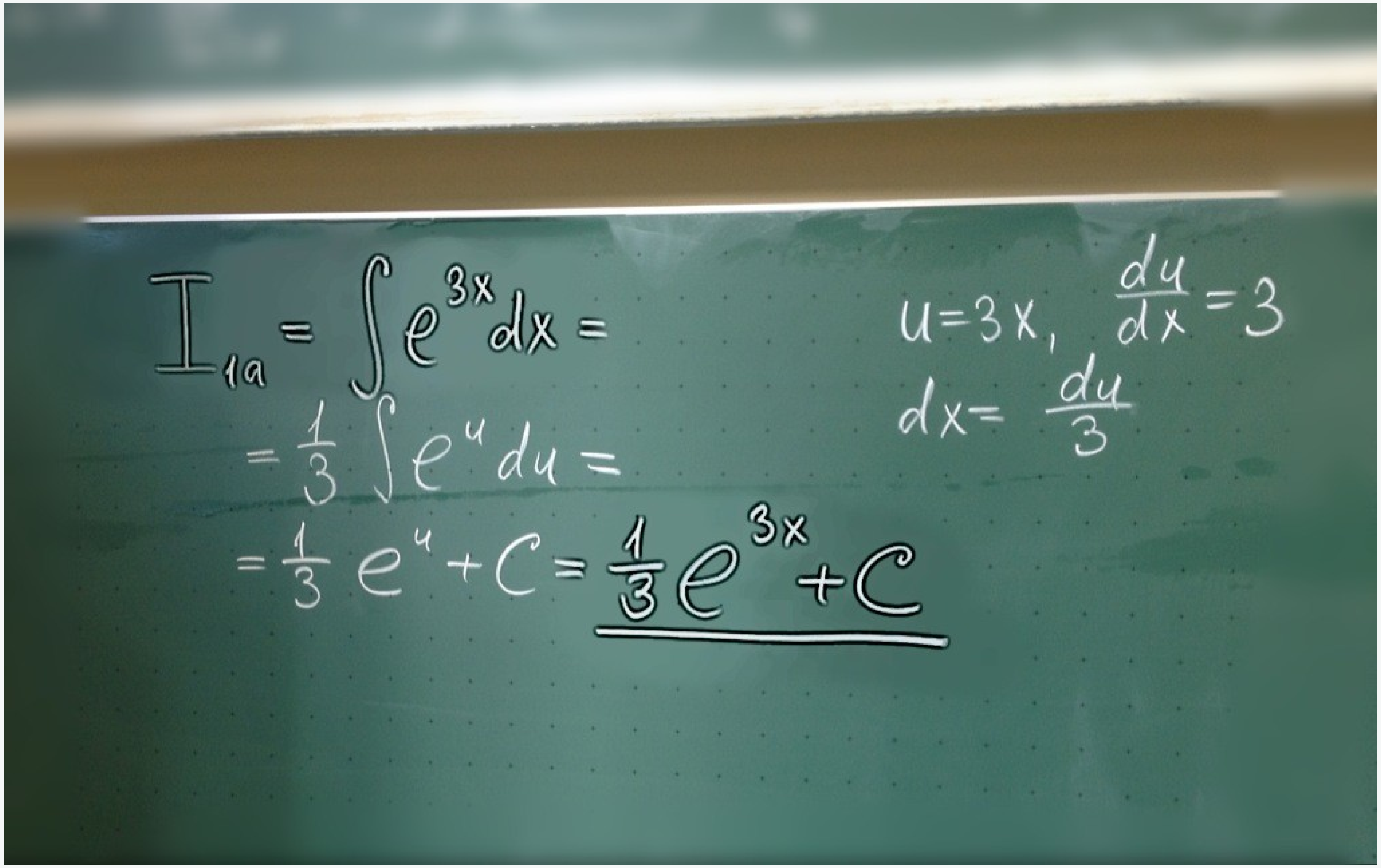
Berechnen Sie folgende unbestimmte Integrale

$$I_{1a} = \int e^{3x}\, dx, \qquad I_{1b} = \int e^{3x-4}\, dx$$

$$I_{2a} = \int x\, e^{2x^2}\, dx, \qquad I_{2b} = \int x^3 e^{5x^4-1}\, dx$$

$$I_{1a} = \int e^{3x}\, dx = \frac{1}{3}\, e^{3x} + C, \qquad u = 3x$$

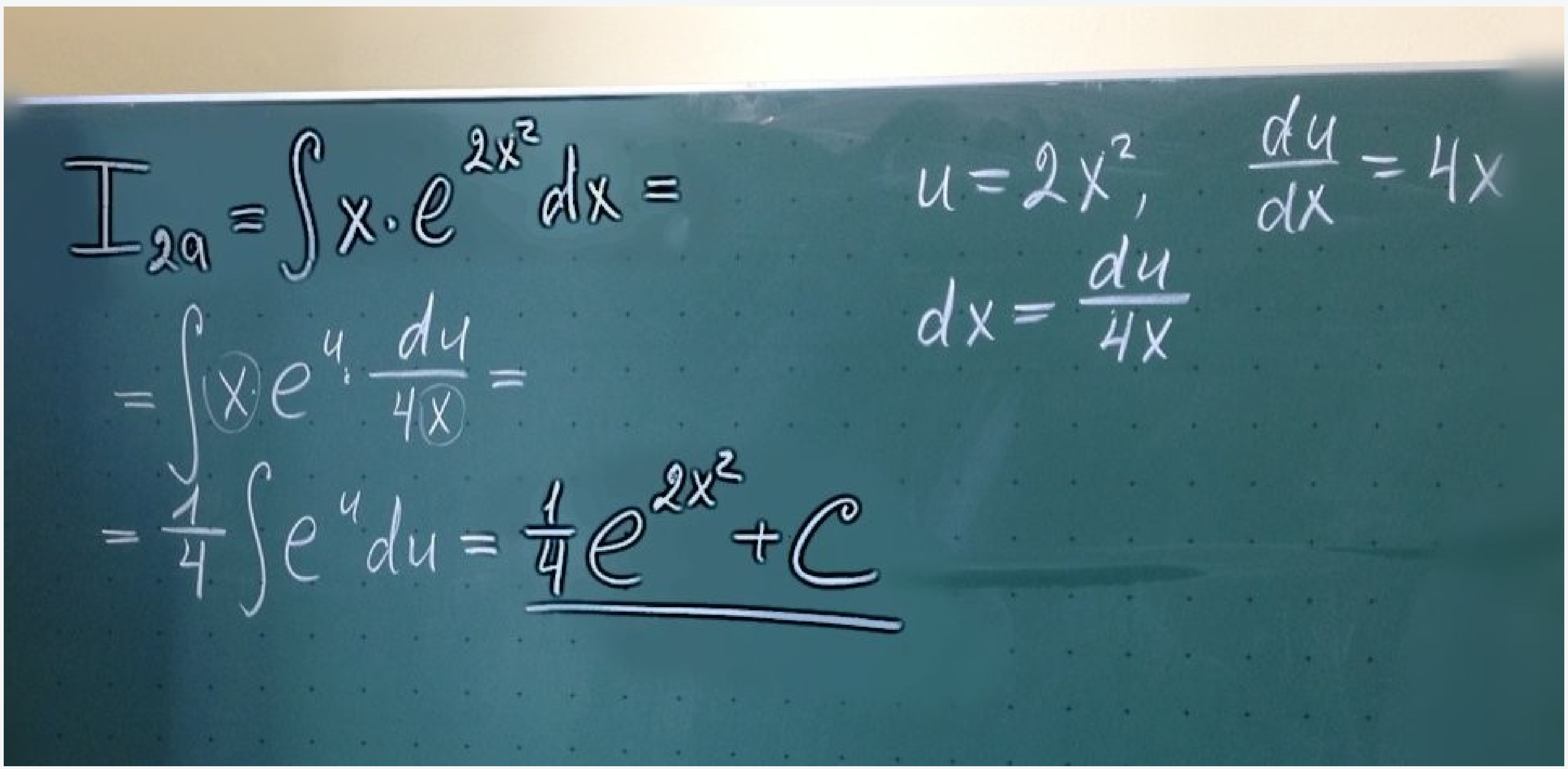
$$I_{1b} = \int e^{3x-4}\, dx = \frac{1}{3}\, e^{3x-4} + C, \qquad u = 3x - 4$$



Die Integration durch Substitution: Lösung 4

$$I_{2a} = \int x\, e^{2x^2}\, dx = \frac{1}{4}\, e^{2x^2} + C, \qquad u = 2x^2$$

$$I_{2b} = \int x^3\, e^{5x^4-1}\, dx = \frac{1}{20}\, e^{5x^4-1} + C, \qquad u = 5x^4 - 1$$



$I_{2a} = \int x \cdot e^{2x^2} dx =$
$= \int x e^u \frac{du}{4x} =$
$= \frac{1}{4} \int e^u du = \frac{1}{4} e^{2x^2} + C$
$u = 2x^2, \quad \frac{du}{dx} = 4x$
$dx = \frac{du}{4x}$
$\int \cos(3x) dx = ?$
$I_{1a} = \int e^{3x} dx =$
$= \frac{1}{3} \int e^u du =$
$= \frac{1}{3} e^u + C = \frac{1}{3} e^{3x} + C$
$u = 3x, \quad \frac{du}{dx} = 3$
$dx = \frac{du}{3}$