Bernoulli-Differentialgleichung: Aufgabe 2

http://www.hnf.de/images/Nautilus_6433.jpg

Lösen Sie folgende Bernoulli-Differenzialgleichung

$$y' + 2\,x\,y = 2\,x\,y^2, \qquad y(0) = 2, \quad y(0) = -2$$

Wir vergleichen die DGL $y' + 2\,x\,y = 2\,x\,y^2$

mit der Bernoulli-Differenzialgleichung $y' + p(x)\,y = q(x)\,y^n$

$$p(x) = 2\,x, \quad q(x) = 2\,x, \quad n = 2 \quad \Rightarrow$$

$$u(x) = y(x)^{1-n} = \frac{1}{y(x)} \quad \Rightarrow \quad u' = -\,\frac{y'}{y^2}$$

$$y' + 2\,x\,y = 2\,x\,y^2 \quad \Big| \quad \frac{1}{y^2}$$

$$\frac{y'}{y^2} + \frac{2\,x}{y} = 2\,x \quad \Rightarrow \quad -u' + 2\,x\,u = 2\,x$$

$$\int \frac{du}{u-1} = 2\int x\,dx \quad \Rightarrow \quad \ln|u-1| = x^2 + \ln|C| \quad \Rightarrow$$

$$\ln\left|\frac{u-1}{C}\right| = x^2 \quad \Rightarrow \quad u = 1 + C\,e^{x^2} \quad \Leftrightarrow \quad \frac{1}{y} = 1 + C\,e^{x^2}$$

Allgemeine Lösung: $$y = \frac{1}{1 + C\, e^{x^2}}$$

Spezielle Lösungen:

$$y(0) = 2, \qquad y = \frac{1}{1 - \frac{1}{2}\, e^{x^2}}, \qquad C = -\frac{1}{2}$$

$$y(0) = -2, \qquad y = \frac{1}{1 - \frac{3}{2}\, e^{x^2}}, \qquad C = -\frac{3}{2}$$

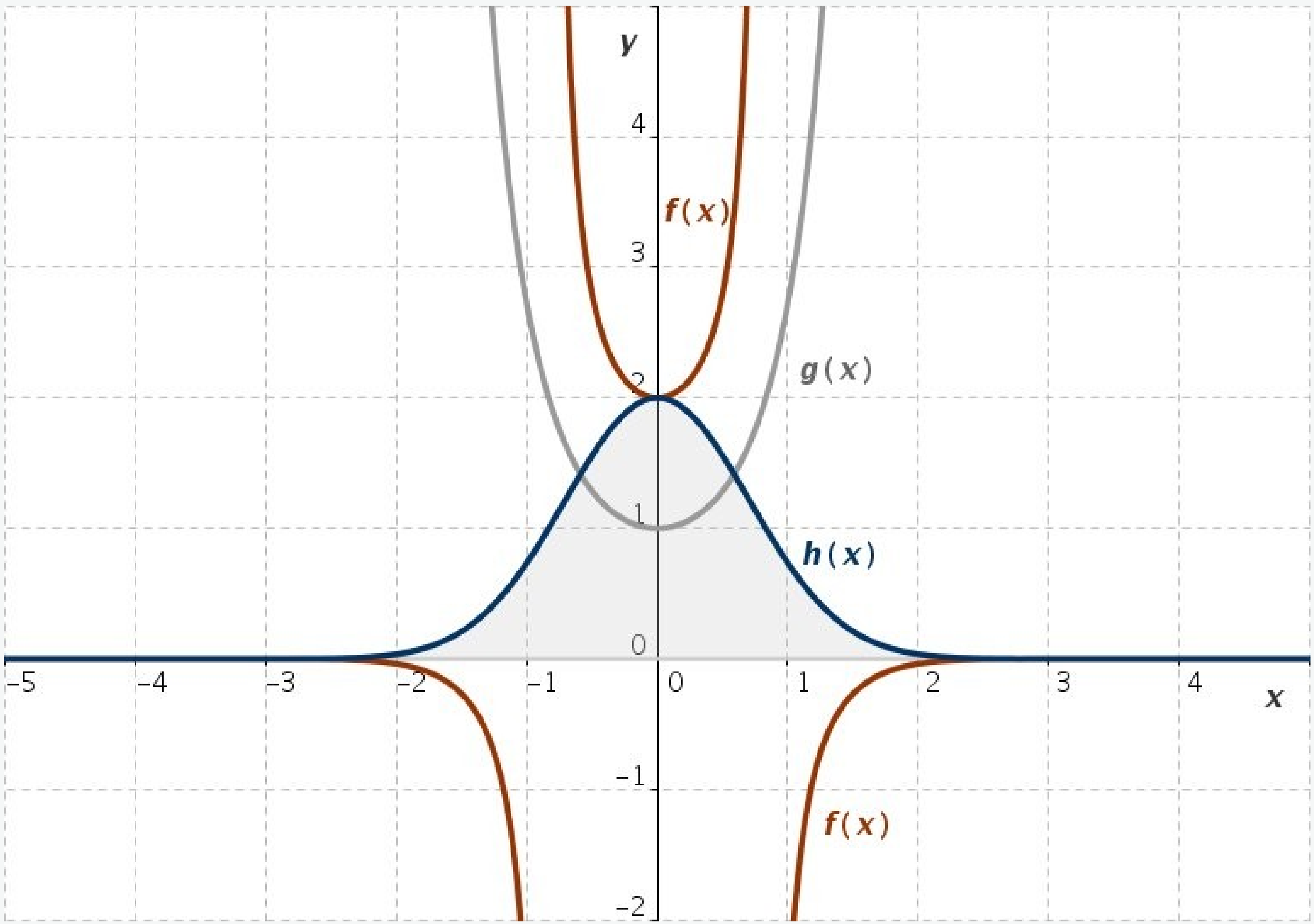

Abb. L2-1: Integralkurve der DGL y = f (x), die C = -1/2 entspricht, Funktionen y = g (x) und y = h (x)

$$f\,(x) = \frac{2}{2 - e^{\,x^2}}\,, \qquad g\,(x) = e^{\,x^2}\,, \qquad h\,(x) = \frac{2}{e^{\,x^2}} = 2\,e^{\,-x^2}$$

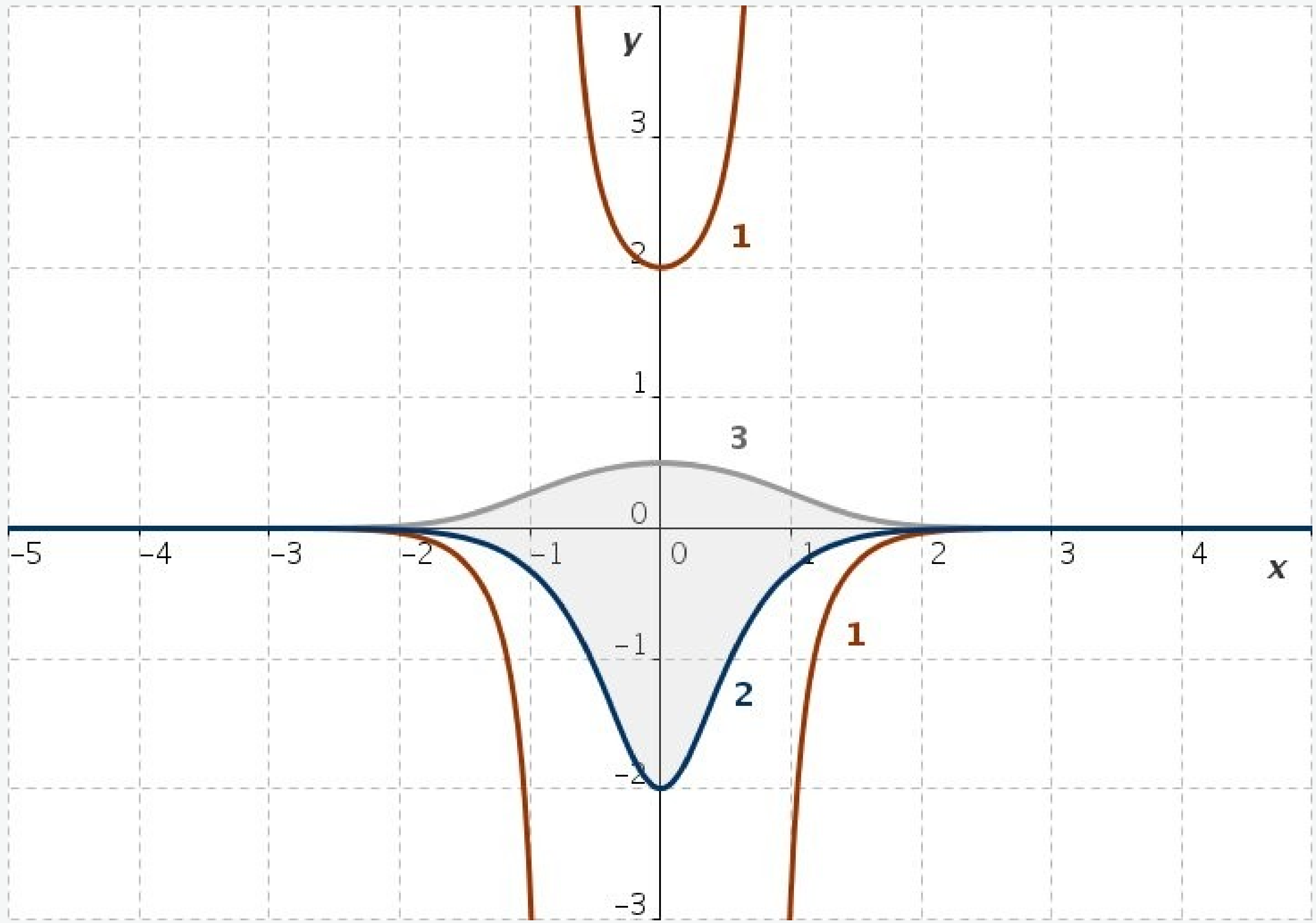

Abb. L2-2: Integralkurven der DGL

$$y = \frac{1}{1 + C\, e^{x^2}}: \qquad 1)\ C = -\frac{1}{2}, \qquad 2)\ C = -\frac{3}{2}, \qquad 3)\ C = 1$$

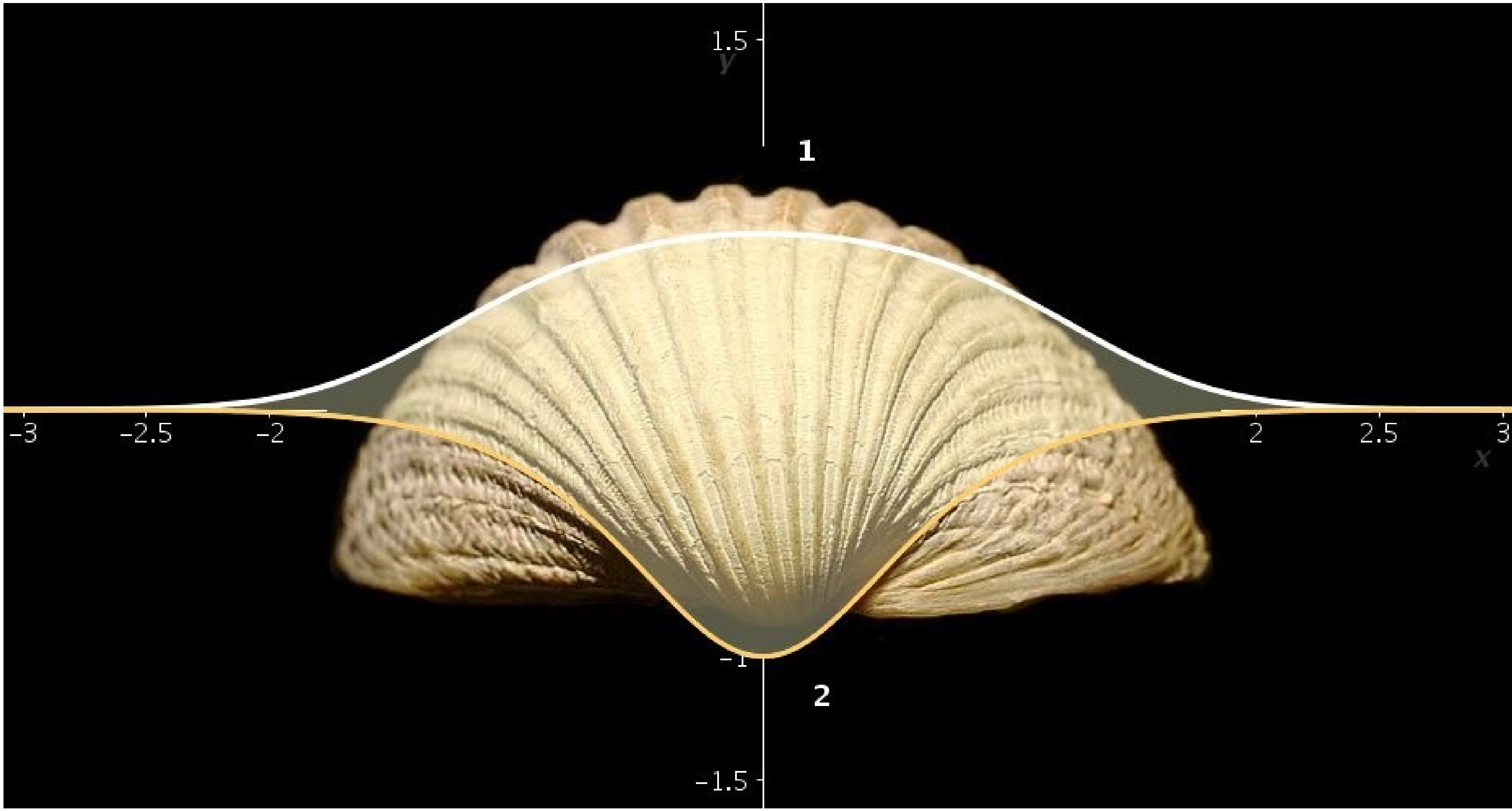

Abb. L2-3: Integralkurven der DGL

$$y = \frac{1}{1 + C\,e^{x^2}}: \qquad 1)\ C = 0.4, \qquad 2)\ C = -2$$