Differentialrechnung: Aufgaben

Jörg Gayler, Lubov Vassilevskaya

Inhaltsverzeichnis

1. Erste Ableitung der elementaren Funktionen

$$C' = 0 \quad (C = const), \qquad (x^n) = n \cdot x^{n-1} \quad (n \in \mathbb{R})$$

$$(\log_a x)' = \frac{1}{x \ln a} \quad (a > 0), \qquad (\ln x)' = \frac{1}{x} \quad (x > 0)$$

$$(e^x)' = e^x, \qquad (a^x)' = (\ln a) \cdot a^x$$

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{\cos^2 x}, \qquad (\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}, \qquad (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$(\sinh x)' = \cosh x, \qquad (\cosh x)' = \sinh x$$

$$(\tanh x)' = \frac{1}{\cosh^2 x}, \qquad (\coth x)' = -\frac{1}{\sinh^2 x}$$

$$(\operatorname{arsinh} x)' = \frac{1}{\sqrt{x^2+1}}, \qquad (\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}}$$

$$(\operatorname{artanh} x)' = \frac{1}{1-x^2}, \qquad (\operatorname{arcoth} x)' = \frac{1}{1-x^2}$$

2. Ableitungsregeln

Faktorregel: $y = C \cdot f(x) \quad (C = const), \qquad y' = C \cdot f'(x)$

Summenregel: $y = f_1(x) + f_2(x) + ... + f_n(x), \qquad y' = f_1'(x) + f_2'(x) + ... + f_n'(x)$

Produktregel: $y = u(x) \cdot v(x), \qquad y' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

Quotientenregel: $y = \dfrac{u(x)}{v(x)}, \qquad y' = \dfrac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v^2(x)}$

Kettenregel: $y = F(u(x)), \qquad y' = \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

3. Einige Regeln

3.1. Rechenregeln mit Potenzen

$$a^n \cdot a^m = a^{n+m}, \qquad \frac{a^n}{a^m} = a^{n-m}, \qquad (a^n)^m = a^{n \cdot m}$$

$$a^n \cdot b^n = (a \cdot b)^n, \qquad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}, \qquad a^n = \frac{1}{a^{-n}}$$

3.2. Rechenregeln mit Wurzeln

Wurzeln als Potenzen mit gebrochenen Exponenten

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \qquad \sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}} = \sqrt[m \cdot n]{a}$$

3.3. Rechenregeln mit Logarithmen

$$\log_a (x \cdot y) = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a (x^r) = r \log_a x$$

$$\log_a \frac{1}{x} = -\log_a x$$

3.4. Trigonometrische Formeln

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin(-\alpha) = -\sin\alpha, \qquad \cos(-\alpha) = \cos\alpha$$

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right), \qquad \cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$$

$$\sin\alpha = \sin(\pi - \alpha), \qquad \cos\alpha = -\cos(\pi - \alpha)$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \qquad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \qquad \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \qquad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right], \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

$$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha), \qquad \cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin(3\alpha)), \qquad \cos^3\alpha = \frac{1}{4}(\cos(3\alpha) + 3\cos\alpha)$$

4. Aufgaben zur Bestimmung von Ableitungen

4.1. Summenregel

Bestimmen Sie die erste Ableitung folgender Funktionen $(a,\, b,\, c\,, m\,, n\, \in\, \mathbb{R})$:

A1

Beispiel:

$$f\,(x) = 2x^3 - \frac{3}{x^2} = 2x^3 - 3\,x^{-2}\,, \qquad f'\,(x) = 2 \cdot 3\,x^2 - 3 \cdot (-2)\,x^{-3} = 6\,x^2 + 6\,x^{-3} = 6\,(x^2 + x^{-3})$$

$$a) \quad f\,(x) = 3x^4 - 2x^3 + 6\,, \qquad g\,(x) = 5x^3 - \frac{x^2}{2} + \frac{5}{x^2}$$

$$b) \quad f\,(x) = \frac{3}{x} + \frac{4}{x^2} - \frac{2}{x^3}\,, \qquad g\,(x) = \frac{x-3}{x^3} + \frac{2x - x^3}{x^4}$$

$$c) \quad f\,(x) = 2\,x^m - 5\,x^n\,, \qquad g\,(x) = a\,x^m + \frac{x^n}{b} - cx^{m-n}$$

A2

Beispiel 1:

$$f\,(x) = x^2 \cdot \sqrt[5]{x} + \frac{1}{x \cdot \sqrt[3]{x}} = x^{2+\frac{1}{5}} + x^{-(1+\frac{1}{3})} = x^{\frac{11}{5}} + x^{-\frac{4}{3}}$$

$$f'\,(x) = \frac{11}{5} \cdot x^{\frac{11}{5}-1} - \frac{4}{3} \cdot x^{-\frac{4}{3}-1} = \frac{11}{5} \cdot x^{\frac{6}{5}} - \frac{4}{3} \cdot x^{-\frac{7}{3}} = \frac{11}{5} \cdot x\,\sqrt[5]{x} - \frac{4}{3\,x^2\,\sqrt[3]{x}}$$

Beispiel 2:

$$f\,(x) = \frac{\sqrt[4]{x}}{\sqrt[3]{x}} = \frac{x^{1/4}}{x^{1/3}} = x^{\frac{1}{4}-\frac{1}{3}} = x^{-\frac{1}{12}}$$

$$f'\,(x) = -\frac{1}{12}\,x^{-\frac{1}{12}-1} = -\frac{1}{12\,x^{1+\frac{1}{12}}} = -\frac{1}{12\,x\,\sqrt[12]{x}}$$

$$a) \quad f\,(x) = x + \sqrt{x}\,, \qquad g\,(x) = x \cdot \sqrt{x} + x \cdot \sqrt[3]{x} + x^2 \cdot \sqrt[3]{x^2}\,, \qquad h\,(x) = \sqrt[3]{x} + \sqrt[5]{x}$$

$$b) \quad f\,(x) = \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{x \cdot \sqrt{x}}\,, \qquad g\,(x) = \frac{1}{\sqrt[3]{x}} + \frac{1}{\sqrt[5]{x^3}} + \frac{1}{\sqrt{x} \cdot \sqrt[6]{x^2}}$$

$$c) \quad f\,(x) = \frac{\sqrt[3]{x}}{\sqrt{x}}\,, \qquad g\,(x) = \frac{\sqrt[3]{x}}{\sqrt[4]{x}}\,, \qquad h\,(x) = \frac{x \cdot \sqrt[3]{x}}{\sqrt[3]{x^2} \cdot \sqrt[6]{x}}$$

A3

Beispiel:

$$f(x) = \sqrt[3]{x^2 \sqrt[3]{x \sqrt[3]{x}}} = \sqrt[3]{x^2 \sqrt[3]{x \cdot x^{1/3}}} = \sqrt[3]{x^2 \sqrt[3]{x^{4/3}}} = \sqrt[3]{x^2 \cdot x^{4/9}} = \sqrt[3]{x^{\frac{22}{9}}} = x^{\frac{22}{27}}$$

$$f'(x) = \frac{22}{27}\, x^{\frac{22}{27}-1} = \frac{22}{27}\, x^{-\frac{5}{27}} = \frac{22}{27\, x^{\frac{5}{27}}} = \frac{22}{27\, \sqrt[27]{x^5}}$$

a) $f(x) = \sqrt{x\sqrt{x}}$, $g(x) = \sqrt{x\sqrt[3]{x}}$, $h(x) = \sqrt[3]{\sqrt{x} \cdot \sqrt[4]{x^2}}$

b) $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$, $g(x) = \sqrt{x\sqrt{x^3\sqrt{x^5}}}$, $h(x) = \sqrt{x\sqrt[3]{x^2\sqrt[4]{x}}}$

A4

a) $f(x) = \sin x + \tan x + \ln x$, $g(x) = 3\sin x - 7\cos x$

b) $f(x) = \tan x - \cot x$, $g(x) = \sinh x + 3\cosh x$

A5 Bestimmen Sie die drei ersten Ableitungen folgender Funktionen:

a) $f(x) = x^4 + x \cdot \sqrt{x} + e^x$

b) $f(x) = \sin x + \cos x$

c) $f(x) = \ln x + e^x$

d) $f(x) = \sinh x + \cosh x$

e) $f(x) = x^n + x^4 + x^3$

4.2. Produktregel

Folgende Funktionen sind unter Verwendung der Produktregel zu differenzieren:

A6

a) $f(x) = (x-2) \cdot (x^2+1)$, $g(x) = (1-x) \cdot (1+x)$

b) $f(x) = (1-x) \cdot (1+x^2+2x^3)$, $g(x) = (1-x) \cdot (1+x^2) \cdot (1+x)$

c) $f(x) = (1+\sqrt{x}) \cdot (1-\sqrt{x})$, $g(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) \cdot (1-\sqrt{x})$

A7

a) $f(x) = x^2 \cdot \ln x\,, \qquad g(x) = (1 + x^2) \cdot \ln x$

b) $f(x) = \sqrt{x} \cdot \cos x\,, \qquad g(x) = x \cdot \cot x\,, \qquad h(x) = x^2 \cdot \tan x$

c) $f(x) = (x^3 + \sqrt{x} + e^x) \cdot \sin x$

4.3. Quotientenregel

Folgende Funktionen sind unter Verwendung der Quotientenregel zu differenzieren:

A8

a) $f(x) = \dfrac{1}{x-1}\,, \qquad g(x) = \dfrac{1}{x+1}\,, \qquad h(x) = \dfrac{3}{5x-1}$

b) $f(x) = \dfrac{x-1}{x+1}\,, \qquad g(x) = \dfrac{x+1}{x-1}\,, \qquad h(x) = \dfrac{x+2}{x-2}$

c) $f(x) = \dfrac{2x-1}{2x+1}\,, \qquad g(x) = \dfrac{2x+1}{2x-1}\,, \qquad h(x) = \dfrac{2x+1}{3x-4}$

d) $f(x) = \dfrac{x}{x^2-1}\,, \qquad g(x) = \dfrac{x}{x^2+1}\,, \qquad h(x) = \dfrac{3x}{4x^2-1}$

e) $f(x) = \dfrac{x^2}{x^2-1}\,, \qquad g(x) = \dfrac{x^2+1}{x^2-1}\,, \qquad h(x) = \dfrac{3x^2+1}{2x^2-4}$

A9

a) $f(x) = \dfrac{1-x}{x^2}\,, \qquad g(x) = \dfrac{1-x^2}{x^3}\,, \qquad h(x) = \dfrac{x-x^3}{x^4}$

b) $f(x) = \dfrac{1}{x^2+2x-1}\,, \qquad g(x) = \dfrac{3x+1}{x^2+2x-1}\,, \qquad h(x) = \dfrac{-x^2+3x+5}{x^2+2x-1}$

c) $f(x) = \dfrac{x^2+2x+1}{x^2-1}\,, \qquad g(x) = \dfrac{x^2+4x+4}{x^2-4}\,, \qquad h(x) = \dfrac{4x^2+12x+9}{4x^2-9}$

d) $f(x) = \dfrac{x^2-6x+9}{2x^2-18}\,, \qquad g(x) = \dfrac{x^2+2x+1}{2x^3-2x}$

e) $f(x) = \dfrac{x^2+2x+1}{(x+1)^3}\,, \qquad g(x) = \dfrac{2x^2-8x+8}{(x-2)^3}$

f) $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}\,, \qquad g(x) = \dfrac{\sqrt{x}\,(1+\sqrt{x})}{1-x}\,, \qquad h(x) = \dfrac{1-x}{\sqrt{x}\,(1+\sqrt{x})}$

g) $f(x) = \dfrac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\,, \qquad g(x) = \dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\,, \qquad h(x) = \dfrac{\sqrt{1-x}}{\sqrt{2+x}}$

4.4. Kettenregel

Folgende Funktionen sind unter Verwendung der Kettenregel zu differenzieren:

A10

a) $f(x) = (x-2)^6\,, \qquad g(x) = (2x+5)^{12}\,, \qquad h(x) = (4x-3)^{-4}$

b) $f(x) = (x^2-4)^5\,, \qquad g(x) = (2x^3-3x^2+6x)^9\,, \qquad h(x) = (4x^{-2}-3)^3$

c) $f(x) = \left(x^2 - \dfrac{4}{x}\right)^4\,, \qquad g(x) = \left(x^3 - \dfrac{3}{x^2}\right)^7\,, \qquad h(x) = \left(x^2 - \dfrac{2}{x} - \dfrac{4}{x^3}\right)^3$

A11

a) $f(x) = \left(\sqrt{x} - x\right)^6\,, \qquad g(x) = \left(\sqrt[3]{x} + 3x\right)^3\,, \qquad h(x) = \left(\sqrt[3]{x+2} + 2x^2 - \ln 2\right)^4$

b) $f(x) = \sqrt{2x+6}\,, \qquad g(x) = \sqrt{x^3 - 2x^2 + 4x}\,, \qquad h(x) = \sqrt[3]{x^3 - 3x}$

c) $f(x) = \sqrt{(2x-3)^3}\,, \qquad g(x) = \sqrt[3]{(x^4+5x+1)^2}\,, \qquad h(x) = \sqrt[5]{x^3 - 1}$

A12

$$f(x) = \left(\frac{x-1}{x+3}\right)^4\,, \qquad g(x) = \left(\frac{x^2+1}{x-3}\right)^5\,, \qquad h(x) = \left(\frac{x^3+3x-2}{x+4}\right)^7$$

A13

$$f(x) = \sqrt{\frac{x+2}{x-2}}\,, \qquad g(x) = \sqrt[5]{\frac{x-1}{x+1}}\,, \qquad h(x) = \sqrt{\frac{x+2}{x^2-4}}$$

A14

$$f(x) = x\cdot\sqrt{x+2}\,, \qquad g(x) = x^2\cdot\sqrt[3]{x-7}\,, \qquad h(x) = (x+2)\cdot\sqrt{x^2-4}$$

A15

a) $f(x) = \sin(3x)\,, \qquad g(x) = \sin(4x+2)\,, \qquad h(x) = \sin(x^2-4)$

b) $f(x) = \cos(-2x)\,, \qquad g(x) = \cos(3x+2)\,, \qquad h(x) = \cos(x^3-4x)$

c) $f(x) = \sin(\sqrt{x})\,, \qquad g(x) = \sin(\sqrt{3x-2})\,, \qquad h(x) = \sin(\sqrt[3]{2x+4})$

d) $f(x) = \cos(\sqrt{x-2})\,, \qquad g(x) = \cos(\sqrt{x^2-5x})\,, \qquad h(x) = \cos(\sqrt[5]{3x+1})$

A16

a) $f(x) = \sqrt{\sin\ x}\,, \quad g(x) = \sqrt{\sin(5x)}\,, \quad h(x) = \sqrt[5]{\sin x}\,, \quad p(x) = \sqrt[5]{\sin(2x-3)}$

b) $f(x) = \sqrt{\cos\ x}\,, \quad g(x) = \sqrt{\cos(2x)}\,, \quad h(x) = \sqrt[4]{\cos x}\,, \quad p(x) = \sqrt[4]{\cos(2x+6)}$

c) $f(x) = \dfrac{1}{\sin(5x)}\,, \quad g(x) = \dfrac{1}{\sqrt{\sin(5x)}}\,, \quad h(x) = \dfrac{\sin\ x}{\sin(5x+2)}\,, \quad p(x) = \dfrac{1}{\sin(x^2)}$

d) $f(x) = \dfrac{1}{\cos(2x)}\,, \quad g(x) = \dfrac{1}{\sqrt{\cos(3x)}}\,, \quad h(x) = \dfrac{\cos\ x}{\sin(3x+1)}\,, \quad p(x) = \dfrac{\sin\ x}{\cos(x^2)}$

A17

a) $f(x) = \sin(x^3)\,, \quad g(x) = \sin^3\ x\,, \quad h(x) = \sin^3(x^2)$

b) $f(x) = \cos(x^4)\,, \quad g(x) = \cos^4\ x\,, \quad h(x) = \cos^4\ x^2$

c) $f(x) = \sin^2(\sqrt{x+2})\,, \quad g(x) = \sqrt{\sin(x^2)+1}\,, \quad h(x) = \sqrt{\sin^3 x + 1}$

d) $f(x) = \cos^2(\sqrt{3x-1})\,, \quad g(x) = \sqrt{\cos(x^3)-7}\,, \quad h(x) = \sqrt{\cos^4 x + 12}$

A18

a) $f(x) = \sin(2x)\cdot\sin(2x+\pi)\,, \quad g(x) = \sin(2x)\cdot\sin(x+\pi)$

b) $f(x) = \cos x\cdot\cos\left(x+\dfrac{\pi}{2}\right)\,, \quad g(x) = \cos(2x)\cdot\cos\left(x+\dfrac{\pi}{2}\right)$

A19

a) $f(x) = \sin^2 x\cdot\cos x\,, \quad g(x) = \sin^2(3x)\cdot\cos(x+\pi)$

b) $f(x) = \sin(x^2)\cdot\cos(x^2)\,, \quad g(x) = \sin(x+2)^2\,, \quad h(x) = \sin(x^2+2)$

c) $f(x) = \sin(x^2)\cdot\cos(x^3)\,, \quad g(x) = \sin(\sqrt{x})\cdot\cos(\sqrt{x})$

A20

$f(x) = e^{2x}\,, \quad g(x) = e^{2x-3}\,, \quad h(x) = e^{2x-x^2}\,, \quad p(x) = e^{2x-x^2+\frac{x^4}{2}}$

A21

a) $f(x) = x^3\cdot e^x\,, \quad g(x) = x^2\cdot e^{2x}\,, \quad h(x) = \dfrac{e^x}{x^3}\,, \quad p(x) = \dfrac{e^x}{\sqrt{x}}$

b) $f(x) = (x-1)\cdot e^x\,, \quad g(x) = (x^2+3)\cdot e^x\,, \quad h(x) = x^2\cdot e^{x^2}\,, \quad p(x) = \dfrac{x^3}{e^x}$

A22

$$a) \quad f(x) = \cos\, x \cdot e^x\,, \qquad g(x) = \cos(3x) \cdot e^{2x}\,, \qquad h(x) = \sin(x^2)\, e^x$$

$$b) \quad f(x) = (\sin(2x) + \cos(2x)) \cdot e^x\,, \qquad g(x) = e^{\sin x}\,, \qquad h(x) = e^{\sin(2x)}$$

4.4.1. Ableitungen von Logarithmusfunktionen

$$\textbf{Regeln:} \qquad (\ln u)' = \frac{u'}{u}\,, \qquad (e^u)' = e^u \cdot u'\,, \qquad (a^u)' = a^u\, \ln a \cdot u' \tag{1}$$

A23

Beispiel 1:

$$f(x) = (2x-1) \cdot \ln(4x+3), \qquad f'(x) = (2x-1)' \cdot \ln(4x+3) + (2x-1) \cdot (\ln(4x+3))'$$

$$u = 4x+3, \qquad (\ln(4x+3))' = (\ln\, u)' = \frac{u'}{u} = \frac{(4x+3)'}{4x+3} = \frac{4}{4x+3}$$

$$f'(x) = 2\, \ln(4x+3) + \frac{4(2x-1)}{4x+3}$$

Beispiel 2:

$$f(x) = \ln(x+2)^3 = 3\, \ln(x+2)\,, \qquad f'(x) = \frac{3}{2+x}$$

$a)$ $f(x) = x \cdot \ln\, x\,, \qquad g(x) = \sqrt{x} \cdot \ln\, x\,, \qquad h(x) = (x^2 + x - 2) \cdot \ln\, x$

$b)$ $f(x) = \ln(5x)\,, \qquad g(x) = x \cdot \ln(3x)\,, \qquad h(x) = x \cdot \ln(3x+1)$

$c)$ $f(x) = (2x+1) \cdot \ln(2x+1)\,, \qquad g(x) = (x^2-1) \cdot \ln(x-1)\,, \qquad h(x) = (x^2-1) \cdot \ln(x^2-1)$

$d)$ $f(x) = \ln(x^2 - 4x + 4)\,, \qquad g(x) = (2x^2 - x) \cdot \ln(2x^2 - x)\,, \qquad h(x) = (x-2) \cdot \ln(x^3 - 8)$

$e)$ $f(x) = \ln(x+4)^2\,, \qquad g(x) = \ln(x-2)^5\,, \qquad h(x) = \ln(x^2 + 3x - 2)^3$

A24

Beispiel 1: $\quad f(x) = \ln(\sqrt[5]{x}) = \ln\left(x^{\frac{1}{5}}\right) = \frac{1}{5}\, \ln\, x\,, \qquad f'(x) = \frac{1}{5x}$

Beispiel 2: $\quad f(x) = \ln(\sqrt{2+x}) = \ln\left((2+x)^{\frac{1}{2}}\right) = \frac{1}{2}\, \ln(2+x), \qquad f'(x) = \frac{1}{2(2+x)}$

$a)$ $f(x) = \ln(\sqrt{x})\,, \qquad g(x) = \ln(\sqrt[3]{x})\,, \qquad h(x) = \ln(\sqrt[n]{x})$

$b)$ $f(x) = \ln(1 + \sqrt{x})\,, \qquad g(x) = \ln(\sqrt{1+x})\,, \qquad h(x) = \ln(1 + \sqrt{1+x})$

$c)$ $f(x) = \ln(\sqrt{x} + \sqrt{1+x})\,, \qquad g(x) = \ln(\sqrt{x} - \sqrt{x-1})$

A25

Beispiel 1:
$$f(x) = \ln\left(\sqrt[3]{\frac{x-2}{x+3}}\right) = \ln\left(\left(\frac{x-2}{x+3}\right)^{\frac{1}{3}}\right) = \frac{1}{3}\ln\left(\frac{x-2}{x+3}\right) = \frac{1}{3}\left(\ln(x-2) - \ln(x+3)\right)$$
$$f'(x) = \frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+3}\right) = \frac{5}{3(x-2)(x+3)}$$

Beispiel 2:
$$f(x) = \ln\left(\frac{1}{\sqrt[3]{9-x^2}}\right) = -\frac{1}{3}\ln(9-x^2) = -\frac{1}{3}\ln\left[(3-x)(3+x)\right] =$$
$$= -\frac{1}{3}\left[\ln(3-x) + \ln(3+x)\right]$$
$$f'(x) = -\frac{1}{3}\left[-\frac{1}{3-x} + \frac{1}{3+x}\right] = \frac{2}{3}\,\frac{x}{9-x^2}$$

a) $f(x) = \ln\left(\frac{x}{x+1}\right)$, $g(x) = \ln\left(\frac{x+2}{x+1}\right)$, $h(x) = \ln\left(\frac{x+2}{x^2-4}\right)$

b) $f(x) = \ln\left(\sqrt{\frac{x}{x+1}}\right)$, $g(x) = \ln\left(\sqrt[3]{\frac{x+2}{x+1}}\right)$, $h(x) = \ln\left(\sqrt[5]{\frac{x+2}{x^2-4}}\right)$

c) $f(x) = \ln\left(\frac{1}{\sqrt{1-x^2}}\right)$, $g(x) = \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$, $h(x) = \ln\left(\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x-1}}\right)$

A26

a) $f(x) = \ln(\cos x)$, $g(x) = \ln(\sin x)$, $h(x) = \ln(\tan x)$

b) $f(x) = \ln(\cos(3x))$, $g(x) = \ln(\sin(2x-5))$, $h(x) = \ln(\tan(2x))$

c) $f(x) = \ln\left(\cos(\sqrt{x})\right)$, $g(x) = \ln\left(\sin(\sqrt{2x-3})\right)$, $h(x) = \ln\left(\tan(\sqrt{x+4})\right)$

d) $f(x) = \ln\left(\cos(\sqrt{x^2-3})\right)$, $g(x) = \ln\left(\sin(\sqrt{x^2-3x})\right)$, $h(x) = \ln\left(\tan(\sqrt{x^2+4})\right)$

A27

a) $f(x) = \ln\left(\sqrt{\cos x}\right)$, $g(x) = \ln\left(\sqrt{2x+\sin x}\right)$, $h(x) = \ln\left(\sqrt{\tan x}\right)$

b) $f(x) = \ln\left(\sqrt[3]{\cos x}\right)$, $g(x) = \ln\left(\sqrt[4]{2x+\sin x}\right)$, $h(x) = \ln\left(\sqrt[3]{\tan x}\right)$

A28

a) $f(x) = \frac{x}{\ln x}$, $g(x) = \frac{x^2-3x}{\ln x}$, $h(x) = \frac{\sqrt{x+2}}{\ln x}$

A29

$f(x) = \frac{1}{x} + \ln x - \frac{\ln x}{x}$, $g(x) = x^2\ln(x^2)$, $h(x) = \ln(\sin x)$

5. Logarithmische Differentiation

Bei manchen Differentiationsaufgaben ist es sinnvoll, die Funktionsgleichung $y = f(x)$ vor dem Differenzieren zu logarithmieren. Das bietet sich immer dann an, wenn sich die Funktionsgleichung durch Anwendung der Logarithmengesetze wesentlich vereinfachen lässt.

A30 Benutzen Sie die Methode der logarithmischen Differentiation, um die Produktregel für die Funktion $f(x) = u(x) \cdot v(x)$ und die Quotientenregel für die Funktion $f(x) = u(x)/v(x)$ zu beweisen.

Bestimmen Sie die Ableitungen folgender Funktionen durch Logarithmische Differentiation

A31

$$a) \quad y = 5^{x}, \qquad y = 3^{4x-2}, \qquad y = 7^{x^3}$$

$$b) \quad y = x^{e^x}, \qquad y = x\, e^{e^x}$$

$$c) \quad y = x\, e^{-x^2}, \qquad y = x^2\, e^{-x}$$

A32

$$a) \quad y = x\, e^{-x}, \qquad y = 4x^3\, e^{-x}$$

$$b) \quad y = (x+2)^{\frac{1}{x}}, \qquad y = (2x^2+7)^{\frac{1}{x}}$$

$$c) \quad y = e^{x^2}\, \frac{x^3\,(x-4)}{x+2}$$

A33

$$a)\ y = (1-x)^2\,(1+x)^4, \qquad b)\ y = (2-x)^{\sqrt{x}}, \qquad c)\ y = \sqrt{x}^{\,\sqrt{x}}$$

A34

Beispiel 1:

$$\begin{aligned}
y &= \frac{1-\sqrt{x}}{1+\sqrt{x}}, \qquad \ln y = \ln(1-\sqrt{x}) - \ln(1+\sqrt{x}) \\
\frac{y'}{y} &= -\frac{1}{2\sqrt{x}}\,\frac{1}{1-\sqrt{x}} - \frac{1}{2\sqrt{x}}\,\frac{1}{1+\sqrt{x}} = -\frac{1}{2\sqrt{x}}\left(\frac{1}{1-\sqrt{x}} + \frac{1}{1+\sqrt{x}}\right) = \\
&= -\frac{1}{2\sqrt{x}}\,\frac{2}{(1-\sqrt{x})(1+\sqrt{x})} = -\frac{1}{\sqrt{x}}\,\frac{1}{(1-x)} \\
y' &= -\frac{y}{\sqrt{x}}\,\frac{1}{(1-x)} = -\frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\,\frac{1}{(1-x)} = -\frac{1}{\sqrt{x}}\,\frac{1}{(1+\sqrt{x})^2} \\
&\quad 1-x = (1-\sqrt{x})(1+\sqrt{x})
\end{aligned}$$

Beispiel 2:

$$
\begin{aligned}
y &= \sqrt{\frac{(x+2)^3\,(x-1)}{x^2+1}} = \left(\frac{(x+2)^3\,(x-1)}{x^2+1}\right)^{\frac{1}{2}} \\
\ln y &= \frac{1}{2}\,\ln\left(\frac{(x+2)^3\,(x-1)}{x^2+1}\right) = \frac{1}{2}\,\left(\ln(x+2)^3 + \ln(x-1) - \ln(x^2+1)\right) = \\
&= \frac{1}{2}\,\left(3\,\ln(x+2) + \ln(x-1) - \ln(x^2+1)\right) \\
(\ln y)' &= \frac{y'}{y} = \frac{1}{2}\left(\frac{3}{x+2} + \frac{1}{x-1} - \frac{2x}{x^2+1}\right) \quad \Rightarrow \\
y' &= \frac{y}{2}\left(\frac{3}{x+2} + \frac{1}{x-1} - \frac{2x}{x^2+1}\right) = \\
&= \frac{1}{2}\left(\frac{3}{x+2} + \frac{1}{x-1} - \frac{2x}{x^2+1}\right)\sqrt{\frac{(x+2)^3\,(x-1)}{x^2+1}}
\end{aligned}
$$

Aufgabe:

a) $y = \dfrac{x-1}{x+1}$, *b)* $y = \dfrac{x+2}{x-2}$, *c)* $y = \dfrac{(x-1)\,(x-2)}{(x+3)\,(x+4)}$, *d)* $y = \sqrt{\dfrac{x+2}{x-2}}$

A35

a) $y = (\sin x)^{x^3}$, *b)* $y = 3x\,(\cos x)^{\frac{x}{2}}$, *c)* $y = \dfrac{\tan x}{e^x}$

6. Ableitungsregeln: Lösungen

6.1. Summenregel

L1

$$a) \quad f(x) = 3x^4 - 2x^3 + 6, \qquad f'(x) = 12x^3 - 6x^2$$

$$g(x) = 5x^3 - \frac{x^2}{2} + \frac{5}{x^2}, \qquad g'(x) = 15x^2 - x - \frac{10}{x^3}$$

$$b) \quad f(x) = \frac{3}{x} + \frac{4}{x^2} - \frac{2}{x^3}, \qquad f'(x) = -\frac{3}{x^2} - \frac{8}{x^3} + \frac{6}{x^4}$$

$$g(x) = \frac{x-3}{x^3} + \frac{2x - x^3}{x^4} = -\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}, \qquad g'(x) = \frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4}$$

$$c) \quad f(x) = 2\,x^m - 5\,x^n, \qquad f'(x) = 2\,m\,x^{m-1} - 5\,n\,x^{n-1}$$

$$g(x) = a\,x^m + \frac{x^n}{b} - cx^{m-n}, \qquad g'(x) = a\,m\,x^{m-1} + \frac{n}{b}\,x^{n-1} + c\,(n-m)\,x^{m-n-1}$$

L2

$$a) \quad f(x) = x + \sqrt{x}, \qquad f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$g(x) = x \cdot \sqrt{x} + x \cdot \sqrt[3]{x} + x^2 \cdot \sqrt[3]{x^2} = x^{3/2} + x^{4/3} + x^{8/3},$$

$$g'(x) = \frac{3}{2}\,x^{1/2} + \frac{4}{3}\,x^{1/3} + \frac{8}{3}\,x^{5/3} = \frac{3}{2}\sqrt{x} + \frac{4}{3}\sqrt[3]{x} + \frac{8}{3}\,x \cdot \sqrt[3]{x^2}$$

$$h(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}, \qquad h'(x) = \frac{1}{3\,x^{2/3}} + \frac{1}{5\,x^{4/5}} = \frac{1}{3\sqrt[3]{x^2}} + \frac{1}{5\sqrt[5]{x^4}}$$

$$b) \quad f(x) = x^{1/2} + x^{-1/2} + x^{-3/2}, \qquad f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\,x\sqrt{x}} - \frac{3}{2\,x^2\sqrt{x}}$$

$$g(x) = \frac{1}{\sqrt[3]{x}} + \frac{1}{\sqrt[5]{x^3}} + \frac{1}{\sqrt{x}\cdot\sqrt[6]{x^2}} = \frac{1}{x^{1/3}} + \frac{1}{x^{3/5}} + \frac{1}{x^{5/6}},$$

$$g'(x) = -\frac{1}{3\,x^{4/3}} - \frac{3}{5\,x^{8/5}} - \frac{5}{6\,x^{11/6}} = -\frac{1}{3\,x\sqrt[3]{x}} - \frac{3}{5\,x\sqrt[5]{x^3}} - \frac{5}{6\,x\sqrt[6]{x^5}}$$

$$c) \quad f(x) = \frac{\sqrt[3]{x}}{\sqrt{x}} = \frac{1}{x^{1/6}}, \qquad f'(x) = -\frac{1}{6\,x^{7/6}} = -\frac{1}{6\,x\sqrt[6]{x}}$$

$$g(x) = \frac{\sqrt[3]{x}}{\sqrt[4]{x}} = x^{1/12}, \qquad g'(x) = \frac{1}{12\,x^{11/12}}$$

$$h(x) = \frac{x\cdot\sqrt[3]{x}}{\sqrt[3]{x^2}\cdot\sqrt[6]{x}} = \sqrt{x}, \qquad h'(x) = \frac{1}{2\sqrt{x}}$$

L3

$$a)\quad f(x) = \sqrt{x\sqrt{x}} = x^{3/4}, \qquad f'(x) = \frac{3}{4\,x^{1/4}} = \frac{3}{4\sqrt[4]{x}}$$

$$g(x) = \sqrt{x\sqrt[3]{x}} = x^{2/3}, \qquad g'(x) = \frac{2}{3\,x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$

$$h(x) = \sqrt[3]{\sqrt{x}\cdot\sqrt[4]{x^2}} = x^{1/3}, \qquad h'(x) = \frac{1}{3\,x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$b)\quad f(x) = \sqrt{x\sqrt{x\sqrt{x}}} = x^{7/8}, \qquad f'(x) = \frac{7}{8\,x^{1/8}} = \frac{7}{8\sqrt[8]{x}}$$

$$g(x) = \sqrt{x\sqrt{x^3\sqrt{x^5}}} = x^{15/8}, \qquad g'(x) = \frac{15}{8}\,x^{7/8} = \frac{15}{8}\sqrt[8]{x^7}$$

$$h(x) = \sqrt{x\sqrt[3]{x^2\sqrt[4]{x}}} = x^{7/8}, \qquad h'(x) = \frac{7}{8\,x^{1/8}} = \frac{7}{8\sqrt[8]{x}}$$

L4

$$a)\quad f(x) = \sin x + \tan x + \ln x, \qquad f'(x) = \cos x + 1 + \tan^2 x + \frac{1}{x}$$

$$g(x) = 3\sin x - 7\cos x, \qquad g'(x) = 3\cos x + 7\sin x$$

$$b)\quad f(x) = \tan x - \cot x, \qquad f'(x) = \frac{1}{\sin^2 x\cdot\cos^2 x}$$

$$g(x) = \sinh x + 3\cosh x, \qquad g'(x) = \cosh x + 3\sinh x$$

L5

$$a)\quad f(x) = x^4 + x^{3/2} + e^x, \qquad f'(x) = 4\,x^3 + \frac{3}{2}\sqrt{x} + e^x$$

$$f''(x) = 12\,x^2 + \frac{3}{4\sqrt{x}} + e^x, \qquad f'''(x) = 24\,x - \frac{3}{8\,x^{3/2}} + e^x$$

$$b)\quad f(x) = \sin x + \cos x, \qquad f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x, \qquad f'''(x) = -\cos x + \sin x$$

$$c)\quad f(x) = \ln x + e^x, \qquad f'(x) = \frac{1}{x} + e^x$$

$$f''(x) = -\frac{1}{x^2} + e^x, \qquad f'''(x) = \frac{2}{x^3} + e^x$$

$$d)\quad f(x) = \sinh x + \cosh x, \qquad f'(x) = f''(x) = f'''(x) = \sinh x + \cosh x$$

$$e)\quad f(x) = x^n + x^4 + x^3, \qquad f'''(x) = n\,(n-1)\,(n-2)\,x^{n-3} + 24\,x + 6$$

6.2. Produktregel

L6

$a)$ $f(x) = (x-2)\cdot(x^2+1)\,,\qquad f'(x) = 1 - 4x + 3x^2$

$g(x) = (1-x)\cdot(1+x) = 1 - x^2\,,\qquad g'(x) = -2x$

$b)$ $f(x) = (1-x)\cdot(1+x^2+2x^3)\,,\qquad f'(x) = -1 + 2x + 3x^2 - 8x^3$

$g(x) = (1-x)\cdot(1+x^2)\cdot(1+x) = (1-x^2)\cdot(1+x^2) = 1 - x^4\,,\qquad g'(x) = -4x^3$

$c)$ $f(x) = (1+\sqrt{x})\cdot(1-\sqrt{x}) = 1 - x\,,\qquad f'(x) = -1$

$$g(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\cdot(1-\sqrt{x})\,,\qquad g'(x) = -1 + \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

L7

$a)$ $f(x) = x^2\cdot\ln x\,,\qquad f'(x) = 2x\,\ln x + x$

$$g(x) = (1+x^2)\,\ln x\,,\qquad g'(x) = 2x\,\ln x + \frac{1+x^2}{x}$$

$b)$ $f(x) = \sqrt{x}\cdot\cos x\,,\qquad f'(x) = \dfrac{\cos x}{2\sqrt{x}} - \sqrt{x}\cdot\sin x$

$g(x) = x\cdot\cot x\,,\qquad g'(x) = \cot x - x\,(1+\cot^2 x) = \cot x - x - x\,\cot^2 x$

$h(x) = x^2\cdot\tan x\,,\qquad h'(x) = 2\,x\,\tan x + x^2\,(1+\tan^2 x)\,,$

$c)$ $f(x) = (x^3 + \sqrt{x} + e^x)\cdot\sin x\,,$

$$f'(x) = \left(3x^2 + \frac{1}{2\sqrt{x}} + e^x\right)\cdot\sin x + \left(x^3 + \sqrt{x} + e^x\right)\cdot\cos x$$

6.3. Quotientenregel

L8

a) $$f(x) = \frac{1}{x-1}\,, \qquad f'(x) = -\frac{1}{(x-1)^2}$$

$$g(x) = \frac{1}{x+1}\,, \qquad g'(x) = -\frac{1}{(x+1)^2}$$

$$h(x) = \frac{3}{5x-1}\,, \qquad h'(x) = -\frac{15}{(5x-1)^2}$$

b) $$f(x) = \frac{x-1}{x+1}\,, \qquad f'(x) = \frac{2}{(x+1)^2}$$

$$g(x) = \frac{x+1}{x-1}\,, \qquad g'(x) = -\frac{2}{(x-1)^2}$$

$$h(x) = \frac{x+2}{x-2}\,, \qquad h'(x) = -\frac{4}{(x-2)^2}$$

c) $$f(x) = \frac{2x-1}{2x+1}\,, \qquad f'(x) = \frac{4}{(2x+1)^2}$$

$$g(x) = \frac{2x+1}{2x-1}\,, \qquad g'(x) = -\frac{4}{(2x-1)^2}$$

$$h(x) = \frac{2x+1}{3x-4}\,, \qquad h'(x) = -\frac{11}{(3x-4)^2}$$

d) $$f(x) = \frac{x}{x^2-1}\,, \qquad f'(x) = -\frac{x^2+1}{(x^2-1)^2}$$

$$g(x) = \frac{x}{x^2+1}\,, \qquad g'(x) = -\frac{x^2-1}{(x^2+1)^2}$$

$$h(x) = \frac{3x}{4x^2-1}\,, \qquad h'(x) = -\frac{3\,(4x^2+1)}{(4x^2-1)^2}$$

e) $$f(x) = \frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} = 1 + \frac{1}{x^2-1}\,, \qquad f'(x) = -\frac{2x}{(x^2-1)^2}$$

$$g(x) = \frac{x^2+1}{x^2-1} = \frac{x^2-1+2}{x^2-1} = 1 + \frac{2}{x^2-1}\,, \qquad g'(x) = -\frac{4x}{(x^2-1)^2}$$

$$h(x) = \frac{3x^2+1}{2x^2-4} = \frac{1}{2}\,\frac{3(x^2-2+2)+1}{x^2-2} = \frac{3}{2} + \frac{7}{2}\,\frac{1}{x^2-2}\,,$$

$$h'(x) = -\frac{7x}{(x^2-2)^2}$$

L9

$$a) \quad f(x) = \frac{1-x}{x^2}, \qquad f'(x) = \frac{x-2}{x^3} = \frac{1}{x^2} - \frac{2}{x^3}$$

$$g(x) = \frac{1-x^2}{x^3}, \qquad g'(x) = \frac{x^2-3}{x^4} = \frac{1}{x^2} - \frac{3}{x^4}$$

$$h(x) = \frac{x-x^3}{x^4}, \qquad h'(x) = \frac{x^2-3}{x^4} = \frac{1}{x^2} - \frac{3}{x^4} = g'(x)$$

Die Ableitungen $f'(x)$, $g'(x)$ und $h'(x)$ kann man auch mit Hilfe der Summenregel bestimmen, indem man die Funktionen $f(x)$, $g(x)$ und $h(x)$ entsprechend darstellt:

$$f(x) = \frac{1}{x^2} - \frac{1}{x}, \qquad g(x) = \frac{1}{x^3} - \frac{1}{x}, \qquad h(x) = \frac{1}{x^3} - \frac{1}{x} = g(x)$$

$$b) \quad f(x) = \frac{1}{x^2+2x-1}, \qquad f'(x) = -\frac{2(x+1)}{(x^2+2x-1)^2}$$

$$g(x) = \frac{3x+1}{x^2+2x-1}, \qquad g'(x) = -\frac{3x^2+2x+5}{(x^2+2x-1)^2}$$

$$h(x) = \frac{-x^2+3x+5}{x^2+2x-1}, \qquad h'(x) = -\frac{5x^2+8x+13}{(x^2+2x-1)^2}$$

$$c) \quad f(x) = \frac{x^2+2x+1}{x^2-1} = \frac{(x+1)^2}{(x-1)(x+1)} = \frac{x+1}{x-1}, \qquad f'(x) = -\frac{2}{(x-1)^2}$$

$$g(x) = \frac{x^2+4x+4}{x^2-4} = \frac{(x+2)^2}{(x-2)(x+2)} = \frac{x+2}{x-2}, \qquad g'(x) = -\frac{4}{(x-2)^2}$$

$$h(x) = \frac{4x^2+12x+9}{4x^2-9} = \frac{(2x+3)^2}{(2x-3)(2x+3)} = \frac{2x+3}{2x-3}, \qquad h'(x) = -\frac{12}{(2x-3)^2}$$

$$d) \quad f(x) = \frac{x^2-6x+9}{2x^2-18} = \frac{(x-3)^2}{2(x^2-9)} = \frac{(x-3)^2}{2(x-3)(x+3)} = \frac{x-3}{2(x+3)}, \qquad f'(x) = \frac{3}{(x+3)^2}$$

$$g(x) = \frac{x^2+2x+1}{2x^3-2x} = \frac{x^2+2x+1}{2x(x^2-1)} = \frac{(x+1)^2}{2x(x-1)(x+1)} = \frac{x+1}{2x(x-1)}$$

$$g'(x) = -\frac{1}{2}\,\frac{x^2+2x-1}{x^2(x-1)^2}$$

$$e) \quad f(x) = \frac{x^2+2x+1}{(x+1)^3} = \frac{(x+1)^2}{(x+1)^3} = \frac{1}{x+1}, \qquad f'(x) = -\frac{1}{(x+1)^2}$$

$$g(x) = \frac{2x^2-8x+8}{(x-2)^3} = \frac{2(x-2)^2}{(x-2)^3} = \frac{2}{x-2}, \qquad g'(x) = -\frac{2}{(x-2)^2}$$

f)
$$f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}, \qquad f'(x) = -\frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}$$
$$g(x) = \frac{\sqrt{x}\,(1+\sqrt{x})}{1-x}, \qquad g'(x) = \frac{1}{2\sqrt{x}\,(1-\sqrt{x})^2}$$
$$h(x) = \frac{1-x}{\sqrt{x}\,(1+\sqrt{x})}, \qquad h'(x) = -\frac{1+2\sqrt{x}+x}{2\,x^{3/2}\,(1+\sqrt{x})^2} = -\frac{1}{2\,x^{3/2}}$$

g)
$$f(x) = \frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}, \qquad f'(x) = -\frac{1}{\sqrt{1+x}\,(1+\sqrt{1+x})^2}$$
$$g(x) = \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}, \qquad g'(x) = \frac{1}{\sqrt{1-x}\,(1+\sqrt{1-x})^2}$$
$$h(x) = \frac{\sqrt{1-x}}{\sqrt{2+x}}, \qquad h'(x) = -\frac{3}{2\sqrt{1-x}\,(2+x)^{3/2}}$$

6.4. Kettenregel

L10

a)
$$f(x) = (x-2)^6, \qquad f'(x) = 6\,(x-2)^5$$
$$g(x) = (2x+5)^{12}, \qquad g'(x) = 24\,(2x+5)^{11}$$
$$h(x) = (4x-3)^{-4}, \qquad h'(x) = -\frac{16}{(4x-3)^5}$$

b)
$$f(x) = (x^2-4)^5, \qquad f'(x) = 10\,x\,(x^2-4)^4$$
$$g(x) = (2x^3-3x^2+6x)^9, \qquad g'(x) = 54\,(2x^3-3x^2+6x)^8\,(x^2-x+1)$$
$$h(x) = (4x^{-2}-3)^3, \qquad h'(x) = -\frac{24}{x^3}\left(\frac{4}{x^2}-3\right)^2 = -\frac{24}{x^7}\left(3x^2-4\right)^2$$

c)
$$f(x) = \left(x^2-\frac{4}{x}\right)^4, \qquad f'(x) = 4\left(x^2-\frac{4}{x}\right)^3\left(2x+\frac{4}{x^2}\right) = \frac{8}{x^5}\,(x^3-4)^3\,(x^3+2)$$
$$g(x) = \left(x^3-\frac{3}{x^2}\right)^7, \qquad g'(x) = 7\left(x^3-\frac{3}{x^2}\right)^6\cdot\left(3x^2+\frac{6}{x^3}\right) = \frac{21}{x^{15}}\,(x^5-3)^6\cdot(x^5+2)$$
$$h(x) = \left(x^2-\frac{2}{x}-\frac{4}{x^3}\right)^3,$$
$$h'(x) = 3\left(x^2-\frac{2}{x}-\frac{4}{x^3}\right)^2\cdot\left(2x+\frac{2}{x^2}+\frac{12}{x^4}\right) = \frac{6}{x^{10}}\,(x^5-2x^2-4)^2\cdot(x^5+x^2+6)$$

L11

a) $$f(x) = \left(\sqrt{x} - x\right)^6, \qquad f'(x) = 6\left(\sqrt{x} - x\right)^5 \cdot \left(\frac{1}{2\sqrt{x}} - 1\right)$$

$$g(x) = \left(\sqrt[3]{x} + 3x\right)^3, \qquad g'(x) = 3\left(x^{1/3} + 3x\right)^2 \cdot \left(\frac{1}{3\,x^{2/3}} + 3\right) = 3\left(\sqrt[3]{x} + 3x\right)^2 \cdot \left(\frac{1}{3\sqrt[3]{x^2}} + 3\right)$$

$$h(x) = \left(\sqrt[3]{x+2} + 2x^2 - \ln 2\right)^4, \qquad h'(x) = 4\left((x+2)^{1/3} + 2x^2 - \ln 2\right)^3 \cdot \left(\frac{1}{3\,(x+2)^{2/3}} + 4x\right)$$

b) $$f(x) = \sqrt{2x+6}, \qquad f'(x) = \frac{1}{\sqrt{2x+6}}$$

$$g(x) = \sqrt{x^3 - 2x^2 + 4x}, \qquad g'(x) = \frac{1}{2}\,\frac{3x^2 - 4x + 4}{\sqrt{x^3 - 2x^2 + 4x}}$$

$$h(x) = \sqrt[3]{x^3 - 3x}, \qquad h'(x) = \frac{x^2 - 1}{(x^3 - 3x)^{2/3}}$$

c) $$f(x) = \sqrt{(2x-3)^3}, \qquad f'(x) = 3\sqrt{2x-3}$$

$$g(x) = \sqrt[3]{(x^4 + 5x + 1)^2}, \qquad g'(x) = \frac{2}{3}\,\frac{4x^3 + 5}{(x^4 + 5x + 1)^{1/3}}$$

$$h(x) = \sqrt[5]{x^3 - 1}, \qquad h'(x) = \frac{3}{5}\,\frac{x^2}{(x^3 - 1)^{4/5}}$$

L12

$$f(x) = \left(\frac{x-1}{x+3}\right)^4, \qquad f'(x) = 16 \cdot \frac{(x-1)^3}{(x+3)^5}$$

$$g(x) = \left(\frac{x^2+1}{x-3}\right)^5, \qquad g'(x) = 5 \cdot \frac{(x^2+1)^4 \cdot (x^2 - 6x - 1)}{(x-3)^6}$$

$$h(x) = \left(\frac{x^3 + 3x - 2}{x+4}\right)^7, \qquad h'(x) = 14 \cdot \frac{(x^3 + 3x - 2)^6 \cdot (x^3 + 6x^2 + 7)}{(x+4)^8}$$

L13

$$f(x) = \sqrt{\frac{x+2}{x-2}}, \qquad f'(x) = -\frac{2 \cdot \sqrt{x-2}}{\sqrt{x+2} \cdot (x-2)^2}$$

$$g(x) = \sqrt[5]{\frac{x-1}{x+1}}, \qquad g'(x) = \frac{2}{5\,(x+1)^2} \cdot \left(\frac{x+1}{x-1}\right)^{4/5}$$

$$h(x) = \sqrt{\frac{x+2}{x^2-4}} = \frac{1}{\sqrt{x-2}}, \qquad h'(x) = -\frac{1}{2\,(x-2)^{3/2}}$$

L14

$$f(x) = x \cdot \sqrt{x+2}\,, \qquad f'(x) = \sqrt{x+2} + \frac{1}{2}\,\frac{x}{\sqrt{x+2}} = \frac{1}{2}\,\frac{3x+4}{\sqrt{x+2}}$$

$$g(x) = x^2 \cdot \sqrt[3]{x-7}\,, \qquad g'(x) = 2x \cdot \sqrt[3]{x-7} + \frac{1}{3} \cdot \frac{x^2}{\sqrt[3]{(x-7)^2}} = \frac{7}{3} \cdot \frac{x\,(x-6)}{(x-7)^{2/3}}$$

$$h(x) = (x+2) \cdot \sqrt{x^2-4}\,, \qquad h'(x) = \sqrt{x^2-4} + \frac{x\,(x+2)}{\sqrt{x^2-4}}$$

L15

a) $f(x) = \sin(3x)\,, \qquad f'(x) = 3\,\cos(3x)$

$g(x) = \sin(4x+2)\,, \qquad g'(x) = 4\,\cos(4x+2)$

$h(x) = \sin(x^2-4)\,, \qquad h'(x) = 2x\,\cos(x^2-4)$

b) $f(x) = \cos(-2x) = \cos(2x)\,, \qquad f'(x) = -2\,\sin(2x)$

$g(x) = \cos(3x+2)\,, \qquad g'(x) = -3\sin(3x+2)$

$h(x) = \cos(x^3-4x)\,, \qquad h'(x) = (4-3x^2)\,\sin(x^3-4x)$

c) $f(x) = \sin(\sqrt{x})\,, \qquad f'(x) = \dfrac{\cos(\sqrt{x})}{2\,\sqrt{x}}$

$g(x) = \sin(\sqrt{3x-2})\,, \qquad g'(x) = \dfrac{3}{2} \cdot \dfrac{\cos(\sqrt{3x-2})}{\sqrt{3x-2}}$

$h(x) = \sin(\sqrt[3]{2x+4})\,, \qquad h'(x) = \dfrac{2}{3} \cdot \dfrac{\cos(\sqrt[3]{2x+4})}{\sqrt[3]{(2x+4)^2}}$

d) $f(x) = \cos(\sqrt{x-2})\,, \qquad f'(x) = -\dfrac{1}{2} \cdot \dfrac{\sin(\sqrt{x-2})}{\sqrt{x-2}}$

$g(x) = \cos(\sqrt{x^2-5x})\,, \qquad g'(x) = -\dfrac{1}{2} \cdot \dfrac{(2x-5)\,\sin(\sqrt{x^2-5x})}{\sqrt{x^2-5x}}$

$h(x) = \cos(\sqrt[5]{3x+1})\,, \qquad h'(x) = -\dfrac{3}{5} \cdot \dfrac{\sin\,\sqrt[5]{3x+1}}{\sqrt[5]{(3x+1)^4}}$

L16

a) $f(x) = \sqrt{\sin x}, \qquad f'(x) = \dfrac{\cos x}{2\sqrt{\sin x}}$

$g(x) = \sqrt{\sin(5x)}, \qquad g'(x) = \dfrac{5}{2}\,\dfrac{\cos(5x)}{\sqrt{\sin(5x)}}$

$h(x) = \sqrt[5]{\sin x}, \qquad h'(x) = \dfrac{1}{5}\,\dfrac{\cos x}{(\sin x)^{4/5}} = \dfrac{1}{5}\,\dfrac{\cos x}{\sqrt[5]{\sin^4 x}}$

$p(x) = \sqrt[5]{\sin(2x-3)}, \qquad p'(x) = \dfrac{2}{5}\,\dfrac{\cos(2x-3)}{(\sin(2x-3))^{4/5}} = \dfrac{2}{5}\,\dfrac{\cos(2x-3)}{\sqrt[5]{\sin^4(2x-3)}}$

b) $f(x) = \sqrt{\cos x}, \qquad f'(x) = -\dfrac{\sin x}{2\sqrt{\cos x}}$

$g(x) = \sqrt{\cos(2x)}, \qquad g'(x) = -\dfrac{\sin(2x)}{\sqrt{\cos(2x)}}$

$h(x) = \sqrt[4]{\cos x}, \qquad h'(x) = -\dfrac{1}{4}\,\dfrac{\sin x}{(\cos x)^{3/4}} = -\dfrac{1}{4}\,\dfrac{\sin x}{\sqrt[4]{\cos^3 x}}$

$p(x) = \sqrt[4]{\cos(2x+6)}, \qquad p'(x) = -\dfrac{1}{2}\,\dfrac{\sin(2x+6)}{(\cos(2x+6))^{3/4}} = -\dfrac{1}{2}\,\dfrac{\sin(2x+6)}{\sqrt[4]{\cos^3(2x+6)}}$

c) $f(x) = \dfrac{1}{\sin(5x)}, \qquad f'(x) = -\dfrac{5\cos(5x)}{\sin^2(5x)}$

$g(x) = \dfrac{1}{\sqrt{\sin(5x)}}, \qquad g'(x) = -\dfrac{5}{2}\,\dfrac{\cos(5x)}{(\sin(5x))^{3/2}} = -\dfrac{5}{2}\,\dfrac{\cos(5x)}{\sqrt{\sin^3(5x)}}$

$h(x) = \dfrac{\sin x}{\sin(5x+2)}, \qquad h'(x) = \dfrac{\cos x}{\sin(5x+2)} - \dfrac{5\sin x\cdot\cos(5x+2)}{\sin^2(5x+2)}$

$p(x) = \dfrac{1}{\sin(x^2)}, \qquad p'(x) = -2x\cdot\dfrac{\cos(x^2)}{\sin^2(x^2)}$

d) $f(x) = \dfrac{1}{\cos(2x)}, \qquad f'(x) = \dfrac{2\sin(2x)}{\cos^2(2x)}$

$g(x) = \dfrac{1}{\sqrt{\cos(3x)}}, \qquad g'(x) = \dfrac{3}{2}\,\dfrac{\sin(3x)}{(\cos(3x))^{3/2}} = \dfrac{3}{2}\,\dfrac{\sin(3x)}{\sqrt{\cos^3(3x)}}$

$h(x) = \dfrac{\cos x}{\sin(3x+1)}, \qquad h'(x) = -\dfrac{\sin x}{\sin(3x+1)} - \dfrac{3\cos x\cdot\cos(3x+1)}{\sin^2(3x+1)}$

$p(x) = \dfrac{\sin x}{\cos(x^2)}, \qquad p'(x) = \dfrac{\cos x}{\cos(x^2)} + \dfrac{2x\cdot\sin x\cdot\sin(x^2)}{\cos^2(x^2)}$

L17

$$a) \quad f(x) = \sin(x^3), \qquad f'(x) = 3x^2 \cos(x^3)$$

$$g(x) = \sin^3 x, \qquad g'(x) = 3 \cos x \cdot \sin^2 x$$

$$h(x) = \sin^3(x^2), \qquad h'(x) = 6x \cdot \cos(x^2) \cdot \sin^2(x^2)$$

$$b) \quad f(x) = \cos(x^4), \qquad f'(x) = -4x^3 \sin(x^4)$$

$$g(x) = \cos^4 x, \qquad g'(x) = -4 \sin x \cdot \cos^3 x$$

$$h(x) = \cos^4(x^2), \qquad h'(x) = -8x \cdot \sin(x^2) \cdot \cos^3(x^2)$$

$$c) \quad f(x) = \sin^2(\sqrt{x+2}), \qquad f'(x) = \frac{\sin(\sqrt{x+2}) \cdot \cos(\sqrt{x+2})}{\sqrt{x+2}}$$

$$g(x) = \sqrt{\sin(x^2) + 1}, \qquad g'(x) = \frac{x \cdot \cos(x^2)}{\sqrt{\sin(x^2) + 1}}$$

$$h(x) = \sqrt{\sin^3 x + 1}, \qquad h'(x) = \frac{3}{2} \frac{\sin^2 x \cdot \cos x}{\sqrt{\sin^3 x + 1}}$$

$$d) \quad f(x) = \cos^2(\sqrt{3x-1}), \qquad f'(x) = -\frac{3 \sin(\sqrt{3x-1}) \cdot \cos(\sqrt{3x-1})}{\sqrt{3x-1}}$$

$$g(x) = \sqrt{\cos(x^3) - 7}, \qquad g'(x) = -\frac{3}{2} \frac{x^2 \cdot \sin(x^3)}{\sqrt{\cos(x^3) - 7}}$$

$$h(x) = \sqrt{\cos^4 x + 12}, \qquad h'(x) = -\frac{2\cos^3 x \cdot \sin x}{\sqrt{\cos^4 x + 12}}$$

L18

$$a) \quad f(x) = \sin(2x) \cdot \sin(2x + \pi) = -\sin^2(2x), \qquad f'(x) = -4\sin(2x) \cdot \cos(2x) = -2\sin(4x)$$

$$g(x) = \sin(2x) \cdot \sin(x + \pi) = -\sin(2x) \cdot \sin x,$$

$$g'(x) = -2\cos(2x) \cdot \sin x - \sin(2x) \cdot \cos x$$

$$b) \quad f(x) = \cos x \cdot \cos\left(x + \frac{\pi}{2}\right) = -\cos x \cdot \sin x = -\frac{1}{2} \sin(2x),$$

$$f'(x) = -\cos(2x) = \sin^2 x - \cos^2 x$$

$$g(x) = \cos(2x) \cdot \cos\left(x + \frac{\pi}{2}\right) = -\cos(2x) \cdot \sin x,$$

$$g'(x) = 2 \sin(2x) \cdot \sin x - \cos(2x) \cos x$$

L19

a) $f(x) = \sin^2 x \cdot \cos x\,,\qquad f'(x) = 2\,\sin x \cdot \cos^2 x \,-\, \sin^3 x = \sin x\,(2\,\cos^2 x \,-\, \sin^2 x)$

$$g(x) = \sin^2(3x) \cdot \cos(x+\pi) = -\sin^2(3x) \cdot \cos x\,,$$

$$g'(x) = -6\sin(3x) \cdot \cos x \cdot \cos(3x) \,+\, \sin^2(3x) \cdot \sin x$$

$$= \sin(3x) \cdot (\sin x \cdot \sin(3x) \,-\, 6\,\cos x \cdot \cos(3x))$$

b) $f(x) = \sin(x^2) \cdot \cos(x^2)\,,\qquad f'(x) = 2x\left(2\,\cos^2(x^2) \,-\, 1\right)$

$$g(x) = \sin((x+2)^2)\,,\qquad g'(x) = 2\,(x+2)\,\cos((x+2)^2)$$

$$h(x) = \sin(x^2+2)\,,\qquad h'(x) = 2\,x\,\cos(x^2+2)$$

c) $f(x) = \sin(x^2) \cdot \cos(x^3)\,,\qquad f'(x) = 2\,x\,\cos(x^2) \cdot \cos(x^3) \,-\, 3\,x^2\,\sin(x^2) \cdot \sin(x^3)$

$$g(x) = \sin(\sqrt{x}) \cdot \cos(\sqrt{x})\,,\qquad g'(x) = \frac{1}{2\,\sqrt{x}}\left(\cos^2(\sqrt{x}) \,-\, \sin^2(\sqrt{x})\right) = \frac{\cos(2\sqrt{x})}{2\,\sqrt{x}}$$

L20

$$f(x) = e^{2x}\,,\qquad f'(x) = 2\,e^{2x}$$

$$g(x) = e^{2x-3}\,,\qquad g'(x) = 2\,e^{2x-3}$$

$$h(x) = e^{2x-x^2}\,,\qquad h'(x) = (2-2x)\,e^{2x-x^2} = 2\,(1-x)\,e^{2x-x^2}$$

$$p(x) = e^{2x-x^2+\frac{x^4}{2}}\,,\qquad p'(x) = 2\,(1-x+x^3)\,e^{2x-x^2+\frac{x^4}{2}}$$

L21

a) $f(x) = x^3 \cdot e^x\,,\qquad f'(x) = x^2\,e^x\,(3+x)$

$$g(x) = x^2 \cdot e^{2x}\,,\qquad g'(x) = 2\,x\,e^{2x}\,(1+x)$$

$$h(x) = \frac{e^x}{x^3}\,,\qquad h'(x) = \frac{e^x}{x^3} - \frac{3\,e^x}{x^4} = \frac{e^x}{x^4}\,(x-3)$$

$$p(x) = \frac{e^x}{\sqrt{x}}\,,\qquad p'(x) = \frac{e^x}{\sqrt{x}} - \frac{1}{2}\,\frac{e^x}{x\,\sqrt{x}} = \frac{e^x}{\sqrt{x}}\left(1 - \frac{1}{2x}\right)$$

b) $f(x) = (x-1) \cdot e^x\,,\qquad f'(x) = x \cdot e^x$

$$g(x) = (x^2+3) \cdot e^x\,,\qquad g'(x) = (x^2+2x+3) \cdot e^x$$

$$h(x) = x^2 \cdot e^{x^2}\,,\qquad h'(x) = 2\,x \cdot e^{x^2}\,(1+x^2)$$

$$p(x) = \frac{x^3}{e^x}\,,\qquad p'(x) = \frac{x^2\,(3-x)}{e^x}$$

L22

a) $f(x) = \cos\, x \cdot e^x\,, \qquad f'(x) = e^x\,(\cos\, x - \sin\, x)$

$g(x) = \cos(3x) \cdot e^{2x}\,, \qquad g'(x) = e^{2x}\,(2\,\cos(3x) - 3\sin(3x))$

$h(x) = \sin(x^2)\, e^x\,, \qquad h'(x) = e^x\,(2x \cdot \cos(x^2) + \sin(x^2))$

b) $f(x) = (\sin(2x) + \cos(2x)) \cdot e^x\,, \qquad f'(x) = e^x\,(3\,\cos(2x) - \sin(2x))$

$g(x) = e^{\sin x}\,, \qquad g'(x) = \cos\, x \cdot e^{\sin x}$

$h(x) = e^{\sin(2x)}\,, \qquad h'(x) = 2\,\cos(2x) \cdot e^{\sin(2x)}$

6.4.1. Ableitungen von Logarithmus- und Exponentialfunktionen

L23

a) $f(x) = x \cdot \ln x\,, \qquad f'(x) = 1 + \ln x$

$g(x) = \sqrt{x} \cdot \ln x\,, \qquad g'(x) = \dfrac{1}{\sqrt{x}} + \dfrac{\ln x}{2\,\sqrt{x}} = \dfrac{2 + \ln x}{2\,\sqrt{x}}$

$h(x) = (x^2 + x - 2) \cdot \ln x\,, \qquad h'(x) = x + 1 - \dfrac{2}{x} + (1 + 2x) \cdot \ln x$

b) $f(x) = \ln(5x)\,, \qquad f'(x) = \dfrac{1}{x}$

$g(x) = x \cdot \ln(3x)\,, \qquad g'(x) = \ln(3x) + 1$

$h(x) = x \cdot \ln(3x + 1)\,, \qquad h'(x) = \ln(3x + 1) + \dfrac{3x}{3x + 1}$

c) $f(x) = (2x + 1) \cdot \ln(2x + 1)\,, \qquad f'(x) = 2 + 2\,\ln(2x + 1) = 2\,[1 + \ln(2x + 1)]$

$g(x) = (x^2 - 1) \cdot \ln(x - 1)\,, \qquad g'(x) = 2\,x\,\ln(x - 1) + x + 1$

$h(x) = (x^2 - 1) \cdot \ln(x^2 - 1)\,, \qquad h'(x) = 2\,x\,\ln(x^2 - 1) + 2x = 2x\,[\ln(x^2 - 1) + 1]$

d) $f(x) = \ln(x^2 - 4x + 4)\,, \qquad f'(x) = \dfrac{2x - 4}{x^2 - 4x + 4} = \dfrac{2}{x - 2}$

$g(x) = (2x^2 - x) \cdot \ln(2x^2 - x)\,, \qquad g'(x) = (4x - 1)\,\ln(2x^2 - x) + 4x - 1$

$h(x) = (x - 2) \cdot \ln(x^3 - 8)\,,$

$h'(x) = \ln(x^3 - 8) + \dfrac{3\,(x - 2)\,x^2}{x^3 - 8} = \ln(x^3 - 8) + \dfrac{3\,x^2}{x^2 + 2x + 4}$

e) $f(x) = \ln(x + 4)^2\,, \qquad f'(x) = \dfrac{2}{x + 4}$

$g(x) = \ln(x - 2)^5\,, \qquad g'(x) = \dfrac{5}{x - 2}$

$h(x) = \ln(x^2 + 3x - 2)^3\,, \qquad h'(x) = \dfrac{3\,(2x + 3)}{x^2 + 3x - 2}$

L24

a) $f(x) = \ln(\sqrt{x}) = \dfrac{1}{2} \ln x, \qquad f'(x) = \dfrac{1}{2x}$

$g(x) = \ln(\sqrt[3]{x}) = \dfrac{1}{3} \ln x, \qquad g'(x) = \dfrac{1}{3x}$

$h(x) = \ln(\sqrt[n]{x}) = \dfrac{1}{n} \ln x, \qquad h'(x) = \dfrac{1}{nx}$

b) $f(x) = \ln(1 + \sqrt{x}), \qquad f'(x) = \dfrac{1}{2\sqrt{x}(1+\sqrt{x})}$

$g(x) = \ln(\sqrt{1+x}), \qquad g'(x) = \dfrac{1}{2(1+x)}$

$h(x) = \ln(1 + \sqrt{1+x}), \qquad h'(x) = \dfrac{1}{2\sqrt{1+x}(1+\sqrt{1+x})}$

c) $f(x) = \ln(\sqrt{x} + \sqrt{1+x}), \qquad f'(x) = \dfrac{\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{1+x}}}{\sqrt{x} + \sqrt{1+x}} = \dfrac{1}{2\sqrt{x}\sqrt{1+x}}$

$g(x) = \ln(\sqrt{x} - \sqrt{x-1}), \qquad g'(x) = -\dfrac{1}{2\sqrt{x}\sqrt{x-1}}$

L25

a) $f(x) = \ln\left(\dfrac{x}{x+1}\right), \qquad f'(x) = \dfrac{1}{x\,(x+1)}$

$g(x) = \ln\left(\dfrac{x+2}{x+1}\right), \qquad g'(x) = -\,\dfrac{1}{(x+1)\,(x+2)}$

$h(x) = \ln\left(\dfrac{x+2}{x^2-4}\right) = \ln\left(\dfrac{x+2}{(x-2)\,(x+2)}\right) = \ln\left(\dfrac{1}{x-2}\right) = \ln\,1\,-\,\ln(x-2) = -\,\ln(x-2)$

$h'(x) = -\,\dfrac{1}{x-2}$

b) $f(x) = \ln\left(\sqrt{\dfrac{x}{x+1}}\right) = \dfrac{1}{2}\,\ln\left(\dfrac{x}{x+1}\right) = \dfrac{1}{2}\,(\ln\,x - \ln(x+1))$

$f'(x) = \dfrac{1}{2}\left(\dfrac{1}{x} - \dfrac{1}{x+1}\right) = \dfrac{1}{2\,x\,(x+1)}$

$g(x) = \ln\left(\sqrt[3]{\dfrac{x+2}{x+1}}\right), \qquad g'(x) = -\,\dfrac{1}{3\,(x+1)\,(x+2)}$

$h(x) = \ln\left(\sqrt[5]{\dfrac{x+2}{x^2-4}}\right), \qquad h'(x) = -\,\dfrac{1}{5\,(x-2)}$

c) $f(x) = \ln\left(\dfrac{1}{\sqrt{1-x^2}}\right), \qquad f'(x) = \dfrac{x}{1-x^2}$

$g(x) = \left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right), \qquad g'(x) = \dfrac{1}{2\,\sqrt{x}\,(1-\sqrt{x})} + \dfrac{1}{2}\,\dfrac{1+\sqrt{x}}{\sqrt{x}\,(1-\sqrt{x})^2} = \dfrac{1}{\sqrt{x}\,(1-\sqrt{x})^2}$

$h(x) = \left(\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x-1}}\right), \qquad h'(x) = \dfrac{1}{2}\,\dfrac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x}\,\sqrt{x-1}\,\sqrt{x+1}}$

L26

a) $f(x) = \ln(\cos x)$, $\quad f'(x) = -\dfrac{\sin x}{\cos x}$,

$g(x) = \ln(\sin x)$, $\quad g'(x) = \dfrac{\cos x}{\sin x}$,

$h(x) = \ln(\tan x)$, $\quad h'(x) = \dfrac{1 + \tan^2 x}{\tan x} = \dfrac{1}{\sin x \cos x}$

b) $f(x) = \ln(\cos(3x))$, $\quad f'(x) = -\dfrac{3\sin(3x)}{\cos(3x)}$,

$g(x) = \ln(\sin(2x-5))$, $\quad g'(x) = \dfrac{2\cos(2x-5)}{\sin(2x-5)}$,

$h(x) = \ln(\tan(2x))$, $\quad h'(x) = \dfrac{2 + 2\tan^2(2x)}{\tan(2x)} = \dfrac{2}{\sin(2x)\cos(2x)}$

c) $f(x) = \ln\left(\cos(\sqrt{x})\right)$, $\quad f'(x) = -\dfrac{1}{2}\,\dfrac{\sin(\sqrt{x})}{\sqrt{x}\cdot\cos(\sqrt{x})}$,

$g(x) = \ln\left(\sin(\sqrt{2x-3})\right)$, $\quad g'(x) = \dfrac{\cos(\sqrt{2x-3})}{\sqrt{2x-3}\cdot\sin(\sqrt{2x-3})}$,

$h(x) = \ln\left(\tan(\sqrt{x+4})\right)$, $\quad h'(x) = \dfrac{1}{2\sqrt{x+4}\,\cos(\sqrt{x+4})\sin(\sqrt{x+4})}$

d) $f(x) = \ln\left(\cos(\sqrt{x^2-3})\right)$, $\quad f'(x) = -\dfrac{x\sin\left(\sqrt{x^2-3}\right)}{\sqrt{x^2-3}\cdot\cos\left(\sqrt{x^2-3}\right)}$,

$g(x) = \ln\left(\sin(\sqrt{x^2-3x})\right)$, $\quad g'(x) = \dfrac{1}{2}\,\dfrac{(2x-3)\cos\left(\sqrt{x^2-3x}\right)}{\sqrt{x^2-3x}\cdot\sin\left(\sqrt{x^2-3x}\right)}$,

$h(x) = \ln\left(\tan(\sqrt{x^2+4})\right)$, $\quad h'(x) = \dfrac{x}{\sqrt{x^2+4}\cdot\cos\left(\sqrt{x^2+4}\right)\cdot\sin\left(\sqrt{x^2+4}\right)}$.

L27

$$a)\quad f(x) = \ln\left(\sqrt{\cos x}\right) = \frac{1}{2}\,\ln\,(\cos x)\,,\qquad f'(x) = -\frac{1}{2}\,\frac{\sin x}{\cos x},$$

$$g(x) = \ln\left(\sqrt{2x+\sin x}\right) = \frac{1}{2}\,\ln\,(2x+\sin x)\,,\qquad g'(x) = \frac{1}{2}\,\frac{2+\cos x}{2x+\sin x}$$

$$h(x) = \ln\left(\sqrt{\tan x}\right) = \frac{1}{2}\,\ln\,(\tan x)\,,\qquad h'(x) = \frac{1}{2\,\sin x\,\cos x}$$

$$b)\quad f(x) = \ln\left(\sqrt[3]{\cos x}\right) = \frac{1}{3}\,\ln\,(\cos x)\,,\qquad f'(x) = -\frac{1}{3}\,\frac{\sin x}{\cos x},$$

$$g(x) = \ln\left(\sqrt[4]{2x+\sin x}\right) = \frac{1}{4}\,\ln\,(2x+\sin x)\,,\qquad g'(x) = \frac{1}{4}\,\frac{2+\cos x}{2x+\sin x}$$

$$h(x) = \ln\left(\sqrt[3]{\tan x}\right) = \frac{1}{3}\,\ln\,(\tan x)\,,\qquad h'(x) = \frac{1}{3}\,\frac{(\tan x)'}{\tan x} = \frac{1}{3}\,\frac{1}{\tan x\,\cos^2 x} =$$

$$= \frac{1}{3}\,\frac{\cos x}{\sin x\,\cos^2 x} = \frac{1}{3\,\sin x\,\cos x} = \frac{2}{3\,\sin(2x)}.$$

L28

$$a)\quad f(x) = \frac{x}{\ln x}\,,\qquad f'(x) = \frac{1}{\ln x} - \frac{1}{\ln^2 x}$$

$$g(x) = \frac{x^2-3x}{\ln x}\,,\qquad g'(x) = \frac{2x-3}{\ln x} - \frac{x-3}{\ln^2 x}$$

$$h(x) = \frac{\sqrt{x+2}}{\ln x}\,,\qquad h'(x) = \frac{1}{2\,\sqrt{x+2}\,\ln x} - \frac{\sqrt{x+2}}{x\,\ln^2 x}$$

L29

$$f(x) = \frac{1}{x} + \ln x - \frac{\ln x}{x}\,,\qquad f'(x) = \frac{-2+x+\ln x}{x^2}$$

$$g(x) = x^2\ln(x^2)\,,\qquad g'(x) = 2x\,(1+\ln(x^2))$$

$$h(x) = \ln(\sin\,x)\,,\qquad h'(x) = \frac{\cos\,x}{\sin\,x} = \cot\,x$$

7. Logarithmische Differentiation

L30

$$f(x) = u(x)\cdot v(x),\qquad \ln\,f = \ln\,u + \ln\,v,\qquad \frac{f'}{f} = \frac{u'}{u} + \frac{v'}{v}$$

$$f' = f\left(\frac{u'}{u} + \frac{v'}{v}\right) = u\,v\left(\frac{u'}{u} + \frac{v'}{v}\right) = u'\,v + v'\,u$$

$$f(x) = \frac{u(x)}{v(x)},\qquad \ln\,f = \ln\,u - \ln\,v,\qquad \frac{f'}{f} = \frac{u'}{u} - \frac{v'}{v}$$

$$f' = f\left(\frac{u'}{u} - \frac{v'}{v}\right) = \frac{u}{v}\left(\frac{u'}{u} - \frac{v'}{v}\right) = \frac{u'}{v} - \frac{uv'}{v^2} = \frac{u'v - v'u}{v^2}$$

L31

$$a)\quad y = 5^{x}, \qquad y' = 5^{x}\ \ln 5, \qquad y = 3^{\,4x-2}, \qquad y' = 4 \cdot 3^{\,4x-2} \ln 3$$

$$y = 7^{x^3}, \qquad y' = 3x^2\, 7^{x^3}\ \ln 7$$

$$b)\quad y = x^{e^x}, \qquad \ln y = e^x\ \ln x, \qquad y' = x^{e^x}\, e^x \left(\frac{1}{x} + \ln x\right)$$

$$y = x\, e^{e^x}, \qquad \ln y = \ln x + e^x, \qquad y' = e^{e^x}\ (1 + x\, e^x)$$

$$c)\quad y = x\, e^{-x^2}, \qquad \ln y = \ln\left(x\, e^{-x^2}\right) = \ln x - x^2$$

$$\frac{y'}{y} = \frac{1}{x} - 2x, \qquad y' = x\, e^{-x^2} \left(\frac{1}{x} - 2x\right) = e^{-x^2}\,(1 - 2x^2)$$

$$y = x^2\, e^{-x}, \qquad y' = x\, e^{-x}\,(2 - x)$$

L32

$$a)\quad y = x\, e^{-x}, \qquad \ln y = \ln x - x, \qquad y' = e^{-x}\,(1 - x)$$

$$y = 4x^3\, e^{-x}, \qquad \ln y = \ln 4 + 3 \ln x - x, \qquad y' = 4x^3\, e^{-x} \left(\frac{3}{x} - 1\right)$$

$$b)\quad y = (x+2)^{\frac{1}{x}}, \qquad \ln y = \frac{1}{x}\ \ln(x+2) = x^{-1}\ \ln(x+2)$$

$$y' = (x+2)^{\frac{1}{x}} \left(-\frac{1}{x^2}\ \ln(x+2)\ +\ \frac{1}{x\,(x+2)}\right)$$

$$y = (2x^2+7)^{\frac{1}{x}}, \qquad \ln y = \frac{1}{x}\ \ln(2x^2+7) = x^{-1}\ \ln(2x^2+7)$$

$$y = (2x^2+7)^{\frac{1}{x}} \left(-\frac{1}{x^2}\ \ln(2x^2+7) + \frac{4}{2x^2+7}\right)$$

$$c)\quad y = e^{x^2}\, \frac{x^3\,(x-4)}{x+2}, \qquad \ln y = \ln\left(e^{x^2}\, \frac{x^3\,(x-4)}{x+2}\right) = x^2 + 3\ \ln x + \ln(x-4) - \ln(x+2)$$

$$\frac{y'}{y} = 2x + \frac{3}{x} + \frac{1}{x-4} - \frac{1}{x+2}, \qquad y' = e^{x^2}\, \frac{x^3\,(x-4)}{x+2} \left(2x + \frac{3}{x} + \frac{1}{x-4} - \frac{1}{x+2}\right)$$

L33

a) $y = (1-x)^2\,(1+x)^4, \qquad \ln\, y = 2\,\ln(1-x) + 4\,\ln(1+x), \qquad \frac{y'}{y} = -\frac{2}{1-x} + \frac{4}{1+x}$

$$y' = (1-x)^2\,(1+x)^4\left(\frac{4}{1+x} - \frac{2}{1-x}\right) = 2\,(1-x)\,(1+x)^3\,(1-3x)$$

b) $y = (2-x)^{\sqrt{x}}, \qquad \ln\, y = \sqrt{x}\,\ln(2-x), \qquad \frac{y'}{y} = \frac{1}{2\sqrt{x}}\,\ln(2-x) - \frac{\sqrt{x}}{2-x}$

$$y' = (2-x)^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\,\ln(2-x) - \frac{\sqrt{x}}{2-x}\right)$$

c) $y = \sqrt{x}^{\,\sqrt{x}}, \qquad \ln\, y = \sqrt{x}\,\ln\,\sqrt{x} = \sqrt{x}\,\ln\,x^{\frac{1}{2}} = \frac{1}{2}\,\sqrt{x}\,\ln\,x$

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}}\left(\frac{1}{2}\,\ln\,x + 1\right), \qquad y' = \frac{\sqrt{x}^{\,\sqrt{x}}}{2\sqrt{x}}\left(\frac{1}{2}\,\ln\,x + 1\right) = \frac{\sqrt{x}^{\,\sqrt{x}-1}}{2}\left(\frac{1}{2}\,\ln\,x + 1\right)$$

L34

a) $y = \frac{x-1}{x+1}, \qquad \ln\, y = \ln(x-1) - \ln(x+1), \qquad \frac{y'}{y} = \frac{1}{x-1} - \frac{1}{x+1}$

$$y' = \frac{x-1}{x+1}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) = \frac{2}{(x+1)^2}$$

b) $y = \frac{x+2}{x-2}, \qquad \ln\, y = \ln(x+2) - \ln(x-2), \qquad \frac{y'}{y} = \frac{1}{x+2} - \frac{1}{x-2}$

$$y' = \frac{x+2}{x-2}\left(\frac{1}{x+2} - \frac{1}{x-2}\right) = -\frac{4}{(x-2)^2}$$

c) $y = \frac{(x-1)\,(x-2)}{(x+3)\,(x+4)}, \qquad \ln\, y = \ln(x-1) + \ln(x-2) - \ln(x+3) - \ln(x+4)$

$$\frac{y'}{y} = \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x+3} - \frac{1}{x+4}, \qquad y' = \frac{(x-1)\,(x-2)}{(x+3)\,(x+4)}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x+3} - \frac{1}{x+4}\right)$$

d) $y = \sqrt{\frac{x+2}{x-2}}, \qquad \ln\, y = \frac{1}{2}\,(\ln(x+2) - \ln(x-2))\,, \qquad \frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x+2} - \frac{1}{x-2}\right)$

$$y' = \frac{1}{2}\,\sqrt{\frac{x+2}{x-2}}\left(\frac{1}{x+2} - \frac{1}{x-2}\right) = -\frac{2\,\sqrt{x-2}}{\sqrt{x+2}\cdot(x-2)^2}$$

L35

a) $y = (\sin x)^{x^3}, \qquad \ln\, y = x^3\, \ln(\sin x)$

$$\frac{y'}{y} = 3x^2\, \ln(\sin x) + x^3\, \frac{\cos x}{\sin x}, \qquad y' = (\sin x)^{x^3} \left(3x^2\, \ln(\sin x) + x^3\, \frac{\cos x}{\sin x}\right)$$

b) $y = 3x\,(\cos x)^{\frac{x}{2}}, \qquad \ln\, y = \ln\, 3 + \ln\, x + \frac{x}{2}\, \ln(\cos x)$

$$y' = 3x\,(\cos x)^{\frac{x}{2}} \left(\frac{1}{x} + \frac{1}{2}\, \ln(\cos x) - \frac{x}{2}\, \tan x\right) = \frac{3}{2}\, x\,(\cos x)^{\frac{x}{2}} \left(\frac{2}{x} + \ln(\cos x) - x\, \tan x\right)$$

c) $y = \dfrac{\tan x}{e^x}, \qquad \ln\, y = \ln(\tan x) - x$

$$\frac{y'}{y} = \frac{1}{\cos^2 x\, \tan x} - 1, \qquad y = \frac{\tan x}{e^x} \left(\frac{1}{\cos^2 x\, \tan x} - 1\right) = e^{-x} \left(\frac{1}{\cos^2 x} - \tan x\right)$$